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Kronecker products for compact semisimple Lie groups

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Abstract. A review is given of the application of S-function techniques to the evaluation of Kronecker products of irreducible representations of compact semisimple Lie groups. Explicit formulae are derived for all irreducible representations of all such groups. Recent developments involving composite Young diagrams are brought to fruition and the vexed problem of SO_{2k} is dealt with completely. New branching rules for the classical groups are given in an appendix. These are exploited in the evaluation of Kronecker products by means of a technique which is applied to both classical and exceptional groups. A discussion is made of various modification rules which are needed to express the final results in standard form.

1. Introduction

The evaluation of Kronecker products of irreducible representations of compact semisimple Lie groups is one of the most frequently carried out tasks in applications of group theory to physics. It forms a necessary prerequisite to many calculations such as those involving the computation of coupling coefficients.

Many different methods have been used to facilitate this evaluation. These methods fall into two main categories: those involving the explicit use of weights and Weyl symmetry operations, and those involving tensors, spinors and Schur function operations.

In the first category Weyl (1939, p 231) described a method applicable to the evaluation of Kronecker products of all irreducible representations of each compact semisimple Lie group. Its use was further recommended by Racah (1964) and Speiser (1964), who emphasised the geometrical nature of the Weyl symmetry operations required for its implementation, and by Biedenharn (1963) and Baird and Biedenharn (1964), who stressed the algebraic aspect of these operations. This extremely effective technique has been assimilated into the physics literature by, amongst others, Judd (1963, p 133) and De Swart (1963).

This method requires a knowledge of the weight multiplicities of just one irreducible representation of the pair whose product is under consideration, and incorporates a sum over the elements of the Weyl symmetry group. This summation can be avoided, as pointed out by Englefield (1981), by using the symmetry properties not only of the formula for the dimensions of an irreducible representation, as first advocated by Baird and Biedenharn (1964), but also of the formulae for the eigenvalues of Casimir operators.

In contrast to this, the explicit use of weight multiplicities may be avoided through the use of the formula due to Steinberg (1961). This expresses the required Kronecker product multiplicities, that is the coefficients in the corresponding Clebsch–Gordan series, in terms of a partition function. The formula involves a double sum over elements of the Weyl symmetry group. The partition function is that introduced by Kostant (1959) in writing down an explicit formula for weight multiplicities.

Such weight multiplicities may also be evaluated by means of the recurrence relations of Freudenthal (1954) and Racah (1964). Once the weight multiplicities are known, other methods of evaluating products become available such as those described by Behrends *et al* (1962), Judd (1963, p 132), Straumann (1965) and Gruber (1970). Useful comments on these methods and their computer implementation have been given in the reviews of Gruber (1973) and Kolman and Beck (1973). It is not our intention to add any further comment other than to state that all these methods involve a separate calculation for each group and that these calculations, tedious even for low-rank semisimple Lie groups, rapidly become unwieldy with increasing rank and do not lend themselves to the derivation of general formulae (Barut and Raczka 1981).

In view of this it is often easier to use an alternative method of evaluating Kronecker products proposed by Patera *et al* (1976). This is based on the fact that not only the dimension, which is just the zeroth-order Dynkin index, but also higher-order Dynkin indices are preserved in the decomposition of a product into its irreducible constituents. As pointed out by Patera (1978) this is sometimes sufficient to determine the irreducible constituents completely, especially if supplemented with information on those constituents having the leading highest weights and a knowledge of the congruence classes of the irreducible representations. The Dynkin indices themselves may be evaluated from a knowledge of the weight vectors and their multiplicities. Thanks to the computer tabulation of the dimensions and higher-order indices by McKay and Patera (1981), many products may be evaluated in this way. Slansky (1981), in particular, has used this tabulation to determine Kronecker products of relevance to the unified gauge theories of elementary particles. He gives many examples for both the classical and the exceptional Lie groups.

However, as with all methods of the first category a separate calculation is needed for each group, there is a severe limitation on the rank of the group and no general formulae emerge. In view of these deficiencies we focus our attention in this paper entirely upon the second category of methods of evaluating Kronecker products: namely those based on tensor and spinor manipulations in which the Schur functions play a pre-eminent role. In doing this the intention is to complete the programme of evaluating all such products using Schur functions, to gather together previous relevant results and to present them all in a consistent format.

This exercise goes back to Weyl (1939, p 127) who pointed out that the decomposition of a tensor space into irreducible subspaces can be accomplished through the use of Young symmetry operators having their origin in the theory of the symmetric group. This provides a connection between the symmetric group and the unitary group which is such that the characters of irreducible representations of the unitary group are nothing other than the symmetric functions, introduced by Schur (1901) and now known as Schur functions (Littlewood 1940, p 84). It follows that the Kronecker product of irreducible representations of the unitary group may be evaluated by means of the beautiful rule of Littlewood and Richardson (1934). This provides an algorithm, involving Young diagrams, for the decomposition of the outer product of Schur functions. It is devoid of any overcounting problems, does not depend on a knowledge of weight multiplicities and is essentially independent of the rank of the unitary group. For these reasons the Littlewood–Richardson rule for the evaluation of unitary group products has been absorbed into the armoury of the practising theoretical physicist through texts such as those of Hamermesh (1962, p 250), Judd (1963, p 136), Lichtenberg (1970, p 111) and Wybourne (1970, p 24).

Similar Schur function methods have been developed by Newell (1951) and Littlewood (1958) for dealing with the Kronecker products of irreducible tensor representations of both the orthogonal and symplectic groups. Furthermore, Brauer and Weyl (1935) gave a complete description of the basic spin irreducible representations of the orthogonal and rotation groups, including an analysis of their Kronecker products. The extension to cover the case of mixed spin–tensor or spinor irreducible representations was initiated by Murnaghan (1938) and Littlewood (1940). Their work allowed all Kronecker products of irreducible representations of the orthogonal and rotation groups to be evaluated (Butler and Wybourne 1969), albeit by a method involving a considerable amount of overcounting. More recently some of this overcounting has been eliminated (King *et al* 1981) but at the cost of complexity in the resulting general formulae.

Fortunately a number of novel generalisations of Young diagram methods have taken place involving composite Young diagrams. Initially this was done in a unitary group context (Abramsky and King 1970, King 1970) but later by Girardi *et al* (1981a, b, 1982) who have given special attention to the orthogonal and symplectic groups. Their results are not always unambiguous and their formulae are rather complex but they did inspire some of the results reported here. At the same time Fischler (1981) has attempted to use Young diagram methods to evaluate Kronecker products of all irreducible representations of the classical groups and to extend these methods to the case of the exceptional groups. Unfortunately his work, which shares the same aim as ours, is fraught with errors and oversimplifications† even for the classical groups. The link between classical and exceptional groups has in addition been thoroughly explored in recent years (Wybourne and Bowick 1977, Wybourne 1979, King and Al Qubanchi 1981a, b, King 1981). In particular, it has been shown that classical Schur function methods are more than somewhat useful in an exceptional group context.

With this preamble we reiterate that our purpose here is to give a general account of the evaluation of Kronecker products of all irreducible representations of each compact semisimple Lie group using Schur function methods. The results are expressed as specific formulae incorporating Schur functions and Schur function series. Their use involves standard Schur function operations and they are free of any ambiguities.

We first outline in § 2 the labelling schemes adopted for irreducible representations. They are chosen to give a natural connection with Schur functions. In the case of the exceptional groups the labels are chosen to be in accord with similar labels used for specific classical subgroups of the same rank.

In carrying out calculations non-standard labels may arise and a complete set of the necessary modification rules are assembled in § 3. They give equivalence relations between a representation with a non-standard label and one with a standard label.

The elementary properties of Schur functions are reviewed in § 4 and applied to the evaluation of Kronecker products for the classical groups in § 5. A method (King

† Prompted by the referee, explicit examples in support of this claim are provided in a series of footnotes making specific reference to the Rules 1, 1a, . . . , 5g given by Fischler (1981).

1981) of exploiting known branching rules to evaluate Kronecker products is outlined in § 6. The relevant classical group branching rules are relegated to an appendix, but are used in § 7 to produce explicit formulae for Kronecker products of the irreducible representations of the classical groups, including the thorniest case of all, namely that of the rotation group, SO_{2k}, in an even-dimensional space. The same method is applied to the exceptional groups in § 8.

Throughout the paper illustrative examples are given, but the culmination of the work is a set of formulae yielding algorithms suitable for the computer evaluation of all Kronecker products. Contemporaneously an interactive program SCHUR, written in Pascal, has been developed which has been used to check all the examples and the efficacy of the formulae derived here.

2. Labelling irreducible representations

Partitions of integers play a key role in labelling the irreducible representations (irreps) of compact semisimple Lie groups. The partition of the positive integer l into p integer parts $\lambda_1, \lambda_2, \ldots, \lambda_p$ with $\lambda_1+\lambda_2+\ldots\lambda_p = l$ and $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_p > 0$ is denoted by $\lambda = (\lambda_1, \lambda_2 \ldots \lambda_p)$. Each such partition λ specifies a regular Young diagram, F^λ, consisting of l boxes arranged in p left-adjusted rows. The length of the ith row is λ_i for $i = 1, 2, \ldots, p$. The partition $\tilde{\lambda}$, conjugate to λ, is defined by $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_q)$ where $\tilde{\lambda}_j$ is the length of the jth column of F^λ for $j = 1, 2, \ldots, q$. The corresponding Young diagram, $F^{\tilde{\lambda}}$, is obtained from F^λ by reflecting in the main diagonal and thus interchanging rows and columns. Clearly $p = \tilde{\lambda}_1$ and $q = \lambda_1$. Furthermore if the length of the main diagonal is r, then the arm lengths, $a_k = \lambda_k - k$, to the right of the diagonal, and the leg lengths, $b_k = \tilde{\lambda}_k - k$, below the diagonal, for $k = 1, 2, \ldots, r$, serve to specify the partition λ in Frobenius notation (Littlewood 1940, p 60)

$$\lambda = \begin{pmatrix} a_1 & a_2 & \ldots & a_r \\ b_1 & b_2 & \ldots & b_r \end{pmatrix}.$$

For example if $\lambda = (6521^2)$ then $\tilde{\lambda} = (532^31)$ and in Frobenius notation $\lambda = \binom{53}{41}$, whilst $p = 5$, $q = 6$, $r = 2$ and $l = 15$. This is made clear in figure 1.

$F^{(6521^2)} =$

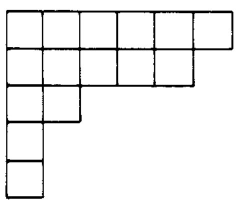


Figure 1.

Irreps of the unitary group, U_n, may be labelled by $\{\lambda\}$ (Littlewood 1940, p 222), where the partition λ serves to specify the symmetry properties of the corresponding lth-rank covariant tensor forming the basis of this representation. This same covariant tensor forms the basis of representations of the subgroups of U_n, including the orthogonal group O_n and, if n is even, the symplectic group Sp_n. However, these representations are, in general, reducible, due to the existence of the symmetric and antisymmetric metric tensors of O_n and Sp_n, respectively. These representations may be reduced by extracting all possible trace terms formed by contraction with the metric tensors. In this way irreps of O_n and Sp_n are obtained, which are conventionally (Littlewood 1940, pp 240, 295) labelled by $[\lambda]$ and $\langle\lambda\rangle$ respectively. The bases are provided by traceless, symmetrised covariant tensors of rank l.

In the case of U_n there also exist inequivalent irreps associated with mth-rank contravariant tensors specified by $\{\bar{\mu}\}$, and more generally irreps associated with mixed tensors specified by $\{\bar{\mu};\lambda\}$. Here the partition λ specifies the symmetry of the l covariant indices and the partition μ specifies the symmetry of the m contravariant indices. The corresponding composite Young diagram, $F^{(\bar{\mu};\lambda)}$, consists of F^{λ} and F^{μ} placed back-to-back (Abramsky and King 1970, King 1970), signifying the tracelessness of the mixed tensor under contraction between covariant and contravariant indices. It is convenient to write $\{\bar{0};\lambda\}=\{\lambda\}$ and $\{\bar{\mu};0\}=\{\bar{\mu}\}$. By way of illustration the composite Young diagram specified by $(\bar{\mu};\lambda)=(\overline{3^221};2^21^4)$ is given in figure 2:

$F^{(\overline{3^221};2^21^4)} =$

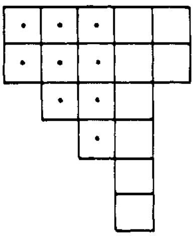


Figure 2.

As well as the tensor irreps labelled by $[\lambda]$, O_n also has double-valued or spinor representations (Brauer and Weyl 1935) denoted by $[\Delta;\lambda]$, where Δ is the fundamental spin representation of dimension $2^{[n/2]}$. The juxtaposition with λ to form $[\Delta;\lambda]$ indicates the leading irreducible component of the Kronecker product of Δ and λ (Littlewood 1950, p 256).

For all linear groups there exists amongst the irreps a one-dimensional irrep, denoted by ε, which maps each group element to the value of its determinant. By definition all the elements of the unimodular groups SU_n, SO_n and Sp_n have determinant $+1$, so that the irreps ε for these groups coincide with the identity irreps $\{0\}$, $[0]$ and $\langle 0\rangle$ respectively. However, for U_n and O_n this is not the case. For U_n ε is the irrep $\{1^n\}$ with an inverse $\varepsilon^{-1}=\bar{\varepsilon}=\{\overline{1^n}\}$. For O_n all group elements have determinant $+1$ or -1. Hence $\varepsilon^{-1}=\varepsilon$ and $\varepsilon\times\varepsilon=[0]$.

The product of ε with any irrep is also an irrep, and inequivalent irreps related by some power of ε are said to be associated. For U_n there are an infinite number of inequivalent irreps associated with a given irrep, one of which will be specified by a partition with less than n parts. For instance $\ldots\{\overline{6^2521}\}$, $\{\overline{5^241}\}$, $\{\overline{4^23};1\}$, $\{\overline{3^22};21\}$, $\{\overline{2^21};32\}$, $\{\overline{1^2};43\}$, $\{541\}$, $\{6521^2\}\ldots$ are all associated irreps of U_5. More generally the Kronecker product $\varepsilon^r\{\bar{\mu};\lambda\}=\varepsilon^r\times\{\bar{\mu};\lambda\}$ is an irrep of U_n associated to the irrep $\{\bar{\mu};\lambda\}$ for any real value of r. If r is not an integer then such irreps are strictly speaking not true irreps of U_n since they are multivalued. In particular, if r is half an odd integer they are analogous to the spinor irreps of O_n, which are of course double valued.

The fact that on restriction from U_n to SU_n $\varepsilon\downarrow\{0\}$ implies that all mutually associated irreps of U_n give equivalent irreps of SU_n under this restriction. Moreover each inequivalent irrep of SU_n may therefore be denoted by means of a partition into less than n parts.

In the case of O_n any given irrep can possess at most one inequivalent associated irrep. Irreps for which the character is zero for all group elements having determinant -1 possess an associate which is equivalent to itself. Such irreps are termed self-associate. For O_{2k} all the spinor irreps and the tensor irreps labelled by partitions having exactly k parts are self-associate. The remaining tensor irreps of O_{2k} and all irreps, spinor and tensor, of O_{2k+1} are not self-associate. Associated pairs of irreps are denoted (Murnaghan 1938, p 276) by $[\lambda]$ and $[\lambda]^* = \varepsilon \times [\lambda]$ and $[\Delta; \lambda]$ and $[\Delta; \lambda]^* = \varepsilon \times [\Delta; \lambda]$.

Under the restriction from O_n to SO_n the distinction between an irrep and its associate is lost. It might be expected that the labels $[\lambda]$ and $[\Delta; \lambda]$ would suffice for irreps of SO_n. This is not the case. Only those irreps of O_n which are not self-associate remain irreducible on restriction from O_n to SO_n. In contrast, each self-associate irrep of O_{2k} yields on restriction to SO_{2k} two inequivalent irreps of the same dimension. These pairs of irreps are conveniently specified by the labels $[\lambda]_\pm$ and $[\Delta; \lambda]_\pm$ where in the former case λ is necessarily a partition into k parts.

The complete list of standard labels for all inequivalent irreps of the classical groups U_n, SU_n, O_n, SO_n and Sp_{2k} is given in table 1.

Table 1. Standard labels for the irreps of classical groups of rank k. $(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ with $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_p > 0$. $(\mu) = (\mu_1, \mu_2, \ldots, \mu_q)$ with $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_q > 0$. λ_i and μ_i are positive integers for $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q$ respectively.

Group G	Label λ_G	Constraint
U_n	$\{\bar{\mu}; \lambda\}$	$p+q \leqslant n = k$
SU_n	$\{\lambda\}$	$p \leqslant n-1 = k$
O_{2k+1}	$[\lambda], [\lambda]^*$	$p \leqslant k$
	$[\Delta; \lambda], [\Delta; \lambda]^*$	$p \leqslant k$
SO_{2k+1}	$[\lambda]$	$p \leqslant k$
	$[\Delta; \lambda]$	$p \leqslant k$
Sp_{2k}	$\langle\lambda\rangle$	$p \leqslant k$
O_{2k}	$[\lambda], [\lambda]^*$	$p < k$
	$[\lambda]$	$p = k$
	$[\Delta; \lambda]$	$p \leqslant k$
SO_{2k}	$[\lambda]$	$p < k$
	$[\lambda]_+, [\lambda]_-$	$p = k$
	$[\Delta; \lambda]_+, [\Delta; \lambda]_-$	$p \leqslant k$

The irreps of the five exceptional groups G_2, F_4, E_6, E_7 and E_8 may conveniently be labelled by exploiting the notation introduced to label the irreps of maximal classical subgroups of the same rank (Wybourne and Bowick 1977, King and Al-Qubanchi 1981a). Many possible maximal embeddings have been considered. Here we choose those embeddings that lead to the greatest simplicity in evaluating Kronecker products. A complete specification of the corresponding standard labels for all inequivalent irreps of the exceptional groups is given in table 2.

The labels introduced here serve to emphasise the tensor or spinor nature of the irreps. It is well known however (Cartan 1894) that each irrep of a compact semisimple Lie group, G, may be labelled by means of its highest weight vector. By working

Table 2. Standard labels for irreps of exceptional groups of rank k. $(\lambda)=(\lambda_1, \lambda_2, \ldots, \lambda_p)$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p > 0$. λ_i is a positive integer for $i = 1, 2, \ldots, p$. s is a positive integer.

Group	Label λ_G		Constraint
G_2	(λ)	$p \leq 2 = k$	$\lambda_1 \geq 2\lambda_2$
F_4	(λ)	$p \leq 4 = k$	$\lambda_1 \geq \lambda_2 + \lambda_3 + \lambda_4$
	$(\Delta; \lambda)$	$p \leq 4 = k$	$\lambda_1 > \lambda_2 + \lambda_3 + \lambda_4$
E_6	$(s:\lambda)$	$p \leq 5 = k-1$	$s \geq \lambda_1 + \lambda_2 + \lambda_3 - \lambda_4 - \lambda_5$
E_7	(λ)	$p \leq 7 = k$	$\lambda_1 \geq \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 - \lambda_6 - \lambda_7$
E_8	(λ)	$p \leq 8 = k$	$\lambda_1 \geq 2\lambda_2 + 2\lambda_3 + 2\lambda_4 - \lambda_5 - \lambda_6 - \lambda_7 - \lambda_8$

with a Euclidean metric in an appropriate weight space and adopting a lexicographic ordering of vectors in that space, it has been demonstrated (King and Al-Qubanchi 1981a) that natural labels $\boldsymbol{\lambda}_G$ may be introduced which are trivially related to the corresponding highest weight vectors. These natural labels $\boldsymbol{\lambda}_G$ are vectors in a k-dimensional space, where k is the rank of the semisimple Lie algebra corresponding to the Lie group G.

In general the natural label $\boldsymbol{\lambda}_G = (\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots, \boldsymbol{\lambda}_k)_G$ is a generalised partition in the sense that $\boldsymbol{\lambda}_1 \geq \boldsymbol{\lambda}_2 \geq \ldots \geq \boldsymbol{\lambda}_k$, without the restriction $\boldsymbol{\lambda}_k > 0$ or even $\boldsymbol{\lambda}_k \geq 0$, whilst $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots, \boldsymbol{\lambda}_k$ are either all integers or all half odd integers. The boldface type serves to distinguish a generalised partition $\boldsymbol{\lambda}$ from an ordinary partition λ.

In tables 1 and 2, quite apart from the use of various brackets $\{\ \}$, $[\ ]$, $\langle\ \rangle$ and $(\ )$ to distinguish different groups rather than the use of the appropriate subscript G, some variations from the natural label notation have been introduced. The correspondence between the standard label λ_G and the natural label $\boldsymbol{\lambda}_G$ is provided by the following identifications:

$$(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_p, 0, 0, \ldots, 0) \tag{2.1a}$$

$$(\bar{\mu}; \lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_p, 0, 0, \ldots, 0, -\mu_q, \ldots, -\mu_2, -\mu_1) \tag{2.1b}$$

$$(\Delta; \lambda) = (\lambda_1 + \tfrac{1}{2}, \lambda_2 + \tfrac{1}{2}, \ldots, \lambda_p + \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}) \tag{2.1c}$$

$$(s; \lambda) = (s, \lambda_1, \lambda_2, \ldots, \lambda_p, 0, 0, \ldots, 0) \tag{2.1d}$$

where each generalised partition $\boldsymbol{\lambda}_G$ has k components if k is the rank of G. For later convenience it is also useful to introduce the following symbols (King *et al* 1981):

$$(\Box; \lambda) = (\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_p + 1, 1, 1, \ldots, 1) \tag{2.2a}$$

$$(\Delta; \bar{\mu}; \lambda) = (\lambda_1 + \tfrac{1}{2}, \lambda_2 + \tfrac{1}{2}, \ldots, -\mu_2 + \tfrac{1}{2}, -\mu_1 + \tfrac{1}{2}) \tag{2.2b}$$

where p and q are the numbers of parts of the partitions λ and μ respectively.

The subscripts $\pm$ used in the case of irreps of SO_{2k} are such that

$$[\lambda]_\pm = [\lambda_1, \lambda_2, \ldots, \lambda_{k-1}, \lambda_k]_\pm = [\lambda_1, \lambda_2, \ldots, \lambda_{k-1}, \pm\lambda_k] \tag{2.3a}$$

$$[\Delta; \lambda]_\pm = [\lambda_1 + \tfrac{1}{2}, \lambda_2 + \tfrac{1}{2}, \ldots, \lambda_{k-1} + \tfrac{1}{2}, \lambda_k + \tfrac{1}{2}]_\pm = [\lambda_1 + \tfrac{1}{2}, \lambda_2 + \tfrac{1}{2}, \ldots, \lambda_{k-1} + \tfrac{1}{2}, \pm\lambda_k \pm \tfrac{1}{2}] \tag{2.3b}$$

$$[\Box; \lambda]_\pm = [\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_{k-1} + 1, \lambda_k + 1]_\pm = [\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_{k-1} + 1, \pm\lambda_k \pm 1] \tag{2.3c}$$

$$[\Delta; \bar{\mu}; \lambda]_\pm = [\lambda_1+\tfrac{1}{2}, \lambda_2+\tfrac{1}{2}, \ldots, -\mu_2+\tfrac{1}{2}, -\mu_1+\tfrac{1}{2}]_\pm$$
$$= [\lambda_1+\tfrac{1}{2}, \lambda_2+\tfrac{1}{2}, \ldots, -\mu_2+\tfrac{1}{2}, \mp\mu_1\pm\tfrac{1}{2}] \tag{2.3d}$$
$$[\bar{\mu}; \lambda]_\pm = [\lambda_1, \lambda_2, \ldots, -\mu_2, -\mu_1]_\pm = [\lambda_1, \lambda_2, \ldots, -\mu_2, \mp\mu_1]. \tag{2.3e}$$

Moreover it should be understood that for any generalised partition $\boldsymbol{\lambda}$

$$[\boldsymbol{\lambda}]_\pm = [\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots, \boldsymbol{\lambda}_{k-1}, \boldsymbol{\lambda}_k]_\pm = [\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots, \boldsymbol{\lambda}_{k-1}, -\boldsymbol{\lambda}_k]_\mp$$
$$= [\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots, \boldsymbol{\lambda}_{k-1}, \pm\boldsymbol{\lambda}_k]_+ = [\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots, \boldsymbol{\lambda}_{k-1}, \mp\boldsymbol{\lambda}_k]_-$$
$$= [\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots, \boldsymbol{\lambda}_{k-1}, \pm\boldsymbol{\lambda}_k]. \tag{2.4}$$

In addition the irreps $[\boldsymbol{\lambda}]_+$ and $[\boldsymbol{\lambda}]_-$ are related by an involutory outer automorphism, $\dagger$, of SO_{2k} which is such that $[\boldsymbol{\lambda}]_\pm^\dagger = [\boldsymbol{\lambda}]_\mp$ for all irrep labels $\boldsymbol{\lambda}$. Thus

$$[\lambda]^\dagger = [\lambda] \qquad \text{if } p<k \tag{2.5a}$$
$$[\lambda]_\pm^\dagger = [\lambda]_\mp \qquad \text{if } p=k \tag{2.5b}$$
$$[\Delta; \lambda]_\pm^\dagger = [\Delta; \lambda]_\mp \qquad \text{if } p\leqslant k. \tag{2.5c}$$

3. Modification rules

Having introduced, in § 2, a complete set of standard labels for all inequivalent irreps of each of the classical and exceptional compact semisimple Lie groups, it should be pointed out that, in almost any attempt to evaluate Kronecker products of irreps, non-standard labels may arise.

For each of the groups G of tables 1 and 2 the label λ_G of an irrep is said to be G-standard if and only if it coincides with one of the tabulated labels. Otherwise λ_G is said to be non-G-standard. In such a case the corresponding character may either vanish identically, or be equal to the character of an irrep specified by a G-standard label, or be the negative of such a character. In the first case the corresponding irrep of G is null and void and we write $\lambda_G = 0$ in recognition of this. In the other two cases we write $\lambda_G = \nu_G$ or $\lambda_G = -\nu_G$ as appropriate, where ν_G is G-standard. Any negative contribution obtained in this way necessarily serves to cancel an identical positive contribution to the complete expression under consideration.

An equality of characters implies an equivalence of representations and the equivalence relations between irreps labelled by non-standard and standard labels are known as modification rules. The simplest of these are those encountered in § 2 as a result of the identification of associated irreps on restriction from a group to a unimodular subgroup. Thus for SU_n we have

$$\{\lambda\} = \{\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n\} = \{\lambda_1-\lambda_n, \lambda_2-\lambda_n, \ldots, \lambda_{n-1}-\lambda_n, 0\} \tag{3.1a}$$
$$\{\bar{\mu}\} = \{-\mu_n, -\mu_{n-1}, \ldots, -\mu_2, -\mu_1\} = \{\mu_1-\mu_n, \mu_1-\mu_{n-1}, \ldots, \mu_1-\mu_2, 0\} \tag{3.1b}$$
$$\{\bar{\mu}; \lambda\} = \{\lambda_1, \lambda_2, \ldots, -\mu_2, -\mu_1\} = \{\mu_1+\lambda_1, \mu_1+\lambda_2, \ldots, \mu_1-\mu_2, 0\} \tag{3.1c}$$

where each vector $\{\ldots\}$ has precisely n components. However, in addition to these modification rules there exists the well known rule:

$$SU_n \qquad \{\lambda\} = \{\lambda_1, \lambda_2, \ldots, \lambda_p\} = 0 \qquad \text{for } p>n. \tag{3.2}$$

This corresponds to the fact that if F^λ contains columns of length greater than n then the associated tensors vanish identically and the irrep $\{\lambda\}$ is null and void.

The first generalisation of this rule was due to Murnaghan (1938, p 282) whose results were considerably extended by Newell (1951). More recently it was realised (King 1971) that all the classical group modification rules could be succinctly expressed in terms of a common procedure. The key operation is that of the removal of a continuous boundary strip of boxes of length h from the Young diagram specified by a partition λ, starting at the foot of the first column and ending in the cth column, to yield symbolically $\lambda - h$. If the resulting Young diagram is regular, in the sense that the columns are non-increasing in length reading from left to right across the diagram, then $\lambda - h$ is simply the partition which serves to specify the diagram. In such a case the relevant sign factor is $(-1)^c$. On the other hand, if the resulting diagram is not regular then it should be discarded since the corresponding character vanishes identically.

By way of illustration if $\lambda = (3^2 21)$ and $h = 3$, 4 or 5 then $\lambda - h = (3^2)$, 0 or (31) respectively with $c = 2$, 2 or 3 as can be seen from figure 3.

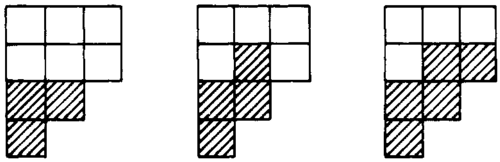


Figure 3.

The modification rules appropriate to the classical groups are summarised in table 3. For the irrep $[6521^2]$ of O_6 with $k = 3$ and $p = 5$, for example, it is easy to see from table 3 that $h = 4$. A glance at figure 1, which illustrates the Young diagram $F^{(65211)}$, shows that a continuous boundary strip of four boxes extends from the foot of the first column into the second column so that $c = 2$. Its removal from $F^{(65211)}$ leaves $F^{(65)}$. Hence for O_6 $[6521^2] = -[65]^*$.

Table 3. Modification rules for the classical groups.

U_n, SU_n	$\{\bar{\mu};\lambda\} = (-1)^{c+d-1}\{\overline{\mu-h};\lambda-h\}$	$h = p+q-n-1 \geqslant 0$
O_{2k+1}	$[\lambda] = (-1)^{c-1}[\lambda-h]^*$	$h = 2p-2k-1 > 0$
	$[\lambda]^* = (-1)^{c-1}[\lambda-h]$	$h = 2p-2k-1 > 0$
	$[\Delta;\lambda] = (-1)^c[\Delta;\lambda-h]^*$	$h = 2p-2k-2 \geqslant 0$
	$[\Delta;\lambda]^* = (-1)^c[\Delta;\lambda-h]$	$h = 2p-2k-2 \geqslant 0$
SO_{2k+1}	$[\lambda] = (-1)^{c-1}[\lambda-h]$	$h = 2p-2k-1 > 0$
	$[\Delta;\lambda] = (-1)^c[\Delta;\lambda-h]$	$h = 2p-2k-2 \geqslant 0$
Sp_{2k}	$\langle\lambda\rangle = (-1)^c\langle\lambda-h\rangle$	$h = 2p-2k-2 \geqslant 0$
O_{2k}	$[\lambda] = (-1)^{c-1}[\lambda-h]^*$	$h = 2p-2k > 0$
	$[\lambda]^* = (-1)^{c-1}[\lambda-h]$	$h = 2p-2k > 0$
	$[\Delta;\lambda] = (-1)^c[\Delta;\lambda-h]$	$h = 2p-2k-1 \geqslant 0$
SO_{2k}	$[\lambda] = (-1)^{c-1}[\lambda-h]$	$h = 2p-2k > 0$
	$[\Delta;\lambda] = (-1)^c[\Delta;\lambda-h]$	$h = 2p-2k-1 \geqslant 0$
	$[\Box;\lambda] = (-1)^{c-1}[\Box;\lambda-h]$	$h = 2p-2k-2 \geqslant 0$
	$[\Delta;\lambda]_\pm = (-1)^c[\Delta;\lambda-h]_\mp$	$h = 2p-2k-1 \geqslant 0$
	$[\Box;\lambda]_\pm = (-1)^{c-1}[\Box;\lambda-h]_\mp$	$h = 2p-2k-2 \geqslant 0$

Similarly for the irrep $\{\overline{3^2 21}; 2^2 1^4\}$ or U_6, $n=6$, $p=6$ and $q=4$ so that $h=3$. From figure 2 it is clear that the removal of continuous boundary strips of length 3 from $F^{(\overline{3321};221111)}$ yields $F^{(\overline{33};221)}$. The two strips extend over one column of $F^{(221111)}$ and two columns of $F^{(3321)}$ so that $c=1$ and $d=2$. Hence for U_6 $\{\overline{3^2 21}; 2^2 1^4\} = \{\overline{3^2}; 2^2 1\}$.

In contrast for U_5 $\{\overline{3^2 21}; 2^2 1^4\} = 0$ since $n=5$, $p=6$ and $q=4$ so that $h=4$, and the removal of a strip of length 4 from $F^{(3321)}$ yields a diagram which is not regular, as can be seen in figure 3.

That the modification rule for U_n and SU_n given in table 3 is a generalisation of (3.2) may be seen by applying the tabulated rule to the irrep $\{\bar{\mu}; \lambda\} = \{\bar{0}; \lambda\} = \{\lambda\}$. In such a case $h = p - n - 1$ so that

$$\{\lambda\} = (-1)^{c+d-1}\{\overline{0-h}; \lambda - h\} = \begin{cases} 0 & \text{for } h>0 \text{ i.e. } p>n+1 \\ -\{\lambda\} = 0 & \text{for } h=0 \text{ i.e. } p=n+1 \end{cases}$$

in agreement with (3.2).

It should be pointed out that the application of the appropriate modification rule of table 3 may not, in one step, convert a non-standard label for an irrep into the corresponding standard label. It may be necessary to repeat the application of the rule. However, it is easy to see that the repeated application of the rules will lead to the restrictions on p and q used in listing the complete set of standard labels for inequivalent irreps of the classical groups in table 1.

The modification rules of table 3 are particularly relevant to calculations carried out using tensor and spinor methods involving, as will be explained in § 4, Schur function manipulations. In contrast to this, a further set of modification rules is needed to cope with certain non-standard labels arising through the use of weight space methods. These modification rules are a direct result of the Weyl reflection symmetry appropriate to weight space (King and Al-Qubanchi 1981b). The key rule of this type is that due originally to Murnaghan (1938, p 132) and Littlewood (1940, p 98), namely

$$(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots, \boldsymbol{\lambda}_i, \boldsymbol{\lambda}_{i+1}, \ldots, \boldsymbol{\lambda}_k) = -(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots, \boldsymbol{\lambda}_{i+1} - 1, \boldsymbol{\lambda}_i + 1, \ldots, \boldsymbol{\lambda}_k) \quad \text{for } i = 1, 2, \ldots, k-1. \tag{3.3}$$

It is this rule whose repeated use allows each inequivalent irrep to be labelled by an ordered sequence taking the form of either a partition or a generalised partition as in tables 1 and 2.

This rule (3.3), although originally formulated for Schur functions in a symmetric group context, is applicable to the natural labels of each of the semisimple Lie groups for $i = 2, 3, \ldots, k-1$ where k is the rank of the group. The case $i = 1$ applies only to the classical groups. The remaining rules which serve to generate all possible weight space modification rules may be obtained from earlier work (King and Al-Qubanchi 1981a). They are presented here in table 4. The particular rule applicable to SU_{k+1} and one of those applicable to each of the groups G_2, E_6, E_7 and E_8 originate in the combination of (3.3) and (3.1*a*).

Finally in the case of the classical groups it is convenient to make repeated use of these rules to derive the composite modification rules of table 5. The notation in this table leans heavily on that of table 3 and involves the formation of $(\overline{\mu - h_1}; \lambda + h)$ from $(\bar{\mu}; \lambda)$ by the removal of a continuous boundary strip of length $h_1 = \mu_1 + \tilde{\mu}_1 - 1$ from F^{μ} to give $F^{\mu - h_1}$, and the addition of a continuous boundary strip of length h

Table 4. Weight space modification rules $\boldsymbol{\lambda}_G = (\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots, \boldsymbol{\lambda}_k) = -\boldsymbol{\nu}_G$, to be used if $h \geq 0$.

Group G	$-\boldsymbol{\nu}_G$	h
All G rank k	$-(\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_i + h, \boldsymbol{\lambda}_{i+1} - h, \ldots, \boldsymbol{\lambda}_k)$	$\boldsymbol{\lambda}_{i+1} - \boldsymbol{\lambda}_i - 1$ for $i = 1, 2, \ldots, k-1$ ($i \neq 1$ for E_6)
SU_{k+1}	$-\{\boldsymbol{\lambda}_1 + h, \boldsymbol{\lambda}_2 + h, \ldots, \boldsymbol{\lambda}_{k-1} + h, \boldsymbol{\lambda}_k + 2h\}$	$-\boldsymbol{\lambda}_k - 1$
SO_{2k+1}	$-[\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots, \boldsymbol{\lambda}_{k-1}, \boldsymbol{\lambda}_k + h]$	$-2\boldsymbol{\lambda}_k - 1$
Sp_{2k}	$-\langle \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots, \boldsymbol{\lambda}_{k-1}, \boldsymbol{\lambda}_k + h \rangle$	$-2\boldsymbol{\lambda}_k - 2$
SO_{2k}	$-[\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots, \boldsymbol{\lambda}_{k-1} + h, \boldsymbol{\lambda}_k + h]$	$-\boldsymbol{\lambda}_{k-1} - \boldsymbol{\lambda}_k - 1$
G_2	$-(\boldsymbol{\lambda}_1 + h, \boldsymbol{\lambda}_2 + 2h)$	$-\boldsymbol{\lambda}_2 - 1$
	$-(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 - h)$	$-\boldsymbol{\lambda}_1 + 2\boldsymbol{\lambda}_2 - 1$
F_4	$-(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3, \boldsymbol{\lambda}_4 + h)$	$-2\boldsymbol{\lambda}_4 - 1$
	$-(\boldsymbol{\lambda}_1 + h, \boldsymbol{\lambda}_2 - h, \boldsymbol{\lambda}_3 - h, \boldsymbol{\lambda}_4 - h)$	$\frac{1}{2}(-\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2 + \boldsymbol{\lambda}_3 + \boldsymbol{\lambda}_4 - 1)$
E_6	$-(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 + h, \boldsymbol{\lambda}_3 + h, \boldsymbol{\lambda}_4 + h, \boldsymbol{\lambda}_5 + h, \boldsymbol{\lambda}_6 + 2h)$	$-\boldsymbol{\lambda}_6 - 1$
	$-(\boldsymbol{\lambda}_1 + h, \boldsymbol{\lambda}_2 - h, \boldsymbol{\lambda}_3 - h, \boldsymbol{\lambda}_4 - h, \boldsymbol{\lambda}_5, \boldsymbol{\lambda}_6)$	$\frac{1}{2}(-\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2 + \boldsymbol{\lambda}_3 + \boldsymbol{\lambda}_4 - \boldsymbol{\lambda}_5 - \boldsymbol{\lambda}_6 - 2)$
E_7	$-(\boldsymbol{\lambda}_1 + h, \boldsymbol{\lambda}_2 + h, \boldsymbol{\lambda}_3 + h, \boldsymbol{\lambda}_4 + h, \boldsymbol{\lambda}_5 + h, \boldsymbol{\lambda}_6 + h, \boldsymbol{\lambda}_7 + 2h)$	$-\boldsymbol{\lambda}_7 - 1$
	$-(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 - h, \boldsymbol{\lambda}_3 - h, \boldsymbol{\lambda}_4 - h, \boldsymbol{\lambda}_5 - h, \boldsymbol{\lambda}_6, \boldsymbol{\lambda}_7)$	$\frac{1}{2}(-\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2 + \boldsymbol{\lambda}_3 + \boldsymbol{\lambda}_4 + \boldsymbol{\lambda}_5 - \boldsymbol{\lambda}_6 - \boldsymbol{\lambda}_7 - 2)$
E_8	$-(\boldsymbol{\lambda}_1 + h, \boldsymbol{\lambda}_2 + h, \boldsymbol{\lambda}_3 + h, \boldsymbol{\lambda}_4 + h, \boldsymbol{\lambda}_5 + h, \boldsymbol{\lambda}_6 + h, \boldsymbol{\lambda}_7 + h, \boldsymbol{\lambda}_8 + 2h)$	$-\boldsymbol{\lambda}_8 - 1$
	$-(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 - h, \boldsymbol{\lambda}_3 - h, \boldsymbol{\lambda}_4 - h, \boldsymbol{\lambda}_5, \boldsymbol{\lambda}_6, \boldsymbol{\lambda}_7, \boldsymbol{\lambda}_8)$	$\frac{1}{3}(-\boldsymbol{\lambda}_1 + 2\boldsymbol{\lambda}_2 + 2\boldsymbol{\lambda}_3 + 2\boldsymbol{\lambda}_4 - \boldsymbol{\lambda}_5 - \boldsymbol{\lambda}_6 - \boldsymbol{\lambda}_7 - \boldsymbol{\lambda}_8)$

Table 5. Composite modification rules.

SO_{2k+1}	$[\bar{\mu};\lambda]=(-1)^r[\overline{\mu-h_1};\lambda+h]$	$h=\mu_1-\tilde{\mu}_1\geqslant 0$
	$[\Delta;\bar{\mu};\lambda]=(-1)^r[\Delta;\overline{\mu-h_1};\lambda+h]$	$h=\mu_1-\tilde{\mu}_1-1\geqslant 0$
Sp_{2k}	$\langle\bar{\mu};\lambda\rangle=(-1)^r\langle\overline{\mu-h_1};\lambda+h\rangle$	$h=\mu_1-\tilde{\mu}_1-1\geqslant 0$
SO_{2k}	$[\bar{\mu};\lambda]_\pm=(-1)^{r-1}[\overline{\mu-h_1};\lambda+h]_\mp$	$h=\mu_1-\tilde{\mu}_1+1\geqslant 0$
	$[\Delta;\bar{\mu};\lambda]_\pm=(-1)^{r-1}[\Delta;\overline{\mu-h_1};\lambda+h]_\mp$	$h=\mu_1-\tilde{\mu}_1\geqslant 0$
$h_1=(\mu_1+\tilde{\mu}_1-1)$		
h extends from the $(k-\tilde{\mu}_1+1)$th row to the $(k-r+1)$th row in $F^{\lambda+h}$.		
Special cases corresponding to $h=0$		
SO_{2k+1}	$[\bar{\varepsilon};\lambda]=(-1)^{(e+r)/2}[\lambda]$	$E=\sum_{\varepsilon}(-1)^{(e+r)/2}\varepsilon$
	$[\Delta;\bar{\gamma};\lambda]=(-1)^{c/2}[\Delta;\lambda]$	$C=\sum_{\gamma}(-1)^{c/2}\gamma$
Sp_{2k}	$\langle\bar{\gamma};\lambda\rangle=(-1)^{c/2}\langle\lambda\rangle$	$C=\sum_{\gamma}(-1)^{c/2}\gamma$
SO_{2k}	$[\bar{\alpha};\lambda]_\pm=(-1)^{a/2}[\lambda]_{\pm(-)^r}$	$A=\sum_{\alpha}(-1)^{a/2}\alpha$
	$[\Delta;\bar{\varepsilon};\lambda]_\pm=(-1)^{(e-r)/2}[\Delta;\lambda]_{\pm(-)^r}$	$G=\sum_{\varepsilon}(-1)^{(e-r)/2}\varepsilon$

to F^λ to give $F^{\lambda+h}$. This addition is the inverse of the removal procedure in the sense that $F^{(\lambda+h)-h}=F^\lambda$.

It is necessarily true that $F^{\mu-h_1}$ is regular. Indeed, in Frobenius notation,

$$\mu=\begin{pmatrix}a_1a_2a_3\ldots a_s\\ b_1b_2b_3\ldots b_s\end{pmatrix}\Rightarrow\mu-h_1=\begin{pmatrix}a_2a_3\ldots a_s\\ b_2b_3\ldots b_s\end{pmatrix}$$

since $h_1=\mu_1+\tilde{\mu}_1-1=a_1+b_1+1$. On the other hand $F^{\lambda+h}$ may not be regular. It is formed by the addition of a continuous boundary strip of length h which must start at its lower end in the first column and extend upwards from the $(k-\tilde{\mu}_1+1)$th row to the $(k-r+1)$th row for the groups SO_{2k+1}, Sp_{2k} and SO_{2k}. If h is negative or if $F^{\lambda+h}$ is not regular then the character of the corresponding irrep is zero and the irrep may be discarded. The modification yields a standard label if it is repeated precisely s times where s is the Frobenius rank of μ.

For example, the irrep $[\overline{62};542]_+$ of SO_{10} is such that $k=5$, $\mu_1=6$ and $\tilde{\mu}_1=2$, so that $h_1=7$, $h=5$ and $k-\tilde{\mu}_1+1=4$. Hence as indicated in figure 4

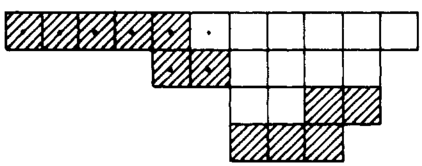


Figure 4.

$(\mu-h_1)=(1)$, $(\lambda+h)=(5443)$ and $r=3$. Thus in SO_{10} $[\overline{62};542]_+=[\bar{1};5443]_-$. Repeating the procedure, or making use of (2.3e), gives $[\bar{1};5443]_-=[54431]_+$ so that the modification rules yield the equivalence relation $[\overline{62}; 542]_+ = [54431]_+$ for SO_{10}.

There are a number of special cases for which at each stage of the modification procedure the crucial parameter h is 0. There are just those cases for which $\mu_i = \tilde{\mu}_i - 1$, $\tilde{\mu}_i$, or $\tilde{\mu}_i + 1$ for $i = 1, 2, \ldots, s$ where s is the Frobenius rank of μ. As is made clear in § 4, such partitions μ are conventionally denoted by α, ε or γ respectively. The corresponding modification rules have been included in table 5.

4. Schur function operations

Partitions are used to label various classical symmetric functions of a set of n indeterminates $x_1, x_2, \ldots, x_n$ (Macdonald 1979, p 10). Amongst these are the Schur functions (S-functions) $s_\lambda = s_\lambda(x_1, x_2, \ldots, x_n)$ (Macdonald 1979, p 23) which were originally introduced by Schur (1901) and exploited to great effect by Littlewood (1940, p 84), who used the notation $\{\lambda\}$ for s_λ.

There is no need for confusion with the notation of § 3 because the character of the irrep $\{\lambda\}$ of U_n is nothing other than the S-function s_λ (Littlewood 1940, p 222). To be precise, the character in the irrep $\{\lambda\}$ of each element of the unitary group U_n belonging to the class labelled by $\boldsymbol{\phi} = (\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_n)$ is given by

$$\chi^{\{\lambda\}}(\boldsymbol{\phi}) = s_{\boldsymbol{\lambda}}(e^{i\boldsymbol{\phi}_1}, e^{i\boldsymbol{\phi}_2}, \ldots, e^{i\boldsymbol{\phi}_n}). \tag{4.1}$$

The indeterminates of s_λ are just the eigenvalues of the group element, which is just an $n \times n$ matrix. This allows for the identification of the irrep $\{\lambda\}$ with its character, namely the S-function s_λ which can thus be denoted by the same symbol $\{\lambda\}$.

This identification reduces the problem of evaluating Kronecker products of irreps of U_n to that of decomposing products of S-functions. This can be done by means of the famous Littlewood–Richardson rule (Littlewood and Richardson 1934, Littlewood 1940, p 34, Macdonald 1979, p 68) which states that

$$\{\lambda\} \cdot \{\mu\} = \sum_\nu m^\nu_{\lambda\mu}\{\nu\} \tag{4.2}$$

where the coefficients $m^\nu_{\lambda\mu}$ are the number of distinctly labelled Young diagrams F^ν obtained from F^λ by the addition of the boxes of F^μ in accordance with the following procedure:

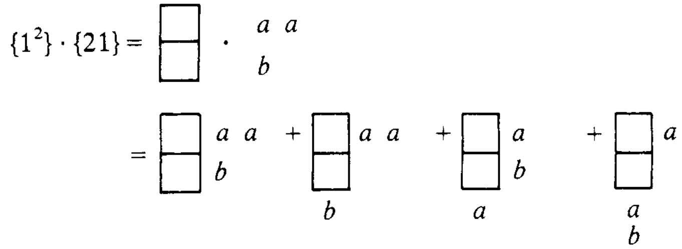
μ_1 letters a, μ_2 letters b, μ_3 letters $c, \ldots$ are added alphabetically to F^λ, one letter at a time in such a way that at every stage:

(i) if the added letters are interpreted as boxes the resulting Young diagram is regular,

(ii) no two identical letters appear in the same column,

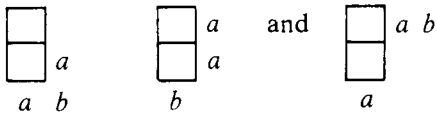
(iii) the sequence of added letters read from right to left across each row in turn from top to bottom is a lattice permutation, in the sense that in this sequence the number of letters $a \geqslant$ the number of letters $b \geqslant$ the number of letters $c \geqslant \ldots$ at every stage of the sequence.

A wealth of literature has grown up around the proof of this theorem (Robinson 1938, Littlewood 1940, p 94, Schutzenberger 1977, Thomas 1978, Macondald 1979, p 68). Many examples of its application are available including a very extensive tabulation due to Butler (Wybourne 1970). It suffices to give just one example here:



$$=\{32\}+\{31^2\}+\{2^21\}+\{21^3\},$$

with other terms such as



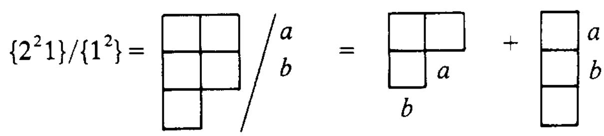
excluded by the rules (i), (ii) and (iii) respectively.

The multiplication operation (4.2) of S-functions serves to define product S-functions $s_{\lambda\cdot\mu}=\{\lambda\cdot\mu\}=\{\lambda\}\cdot\{\mu\}$. The formula (4.2) corresponds to the mutual symmetrisation of two sets of covariant indices, each of well defined symmetry specified by λ and μ, to give tensors with indices of symmetry ν.

A related operation is that of the contraction of one set of contravariant indices of symmetry μ with a subset of a set of covariant tensor indices of symmetry ν to yield covariant tensors with indices of symmetry λ. The corresponding operation on S-functions is that of division, which defines (Littlewood 1940, p 110, Macdonald 1979, p 39) skew S-functions $s_{\nu/\mu}=\{\nu/\mu\}=\{\nu\}/\{\mu\}$ which may be expanded in accordance with the formula

$$\{\nu\}/\{\mu\}=\sum_{\lambda} m^{\nu}_{\lambda\mu}\{\lambda\}. \tag{4.3}$$

The coefficients are nothing other than the Littlewood–Richardson coefficients appearing in (4.2) (Stanley 1971, Lascoux 1977, Schutzenberger 1977). Thus $\{\nu/\mu\}$ is the sum of those $\{\lambda\}$ whose product with $\{\mu\}$ yields $\{\nu\}$. For example



$$=\{21\}+\{1^3\}.$$

It should be pointed out that by virtue of the Littlewood–Richardson rule $m^{\nu}_{\lambda\mu}>0$ only if $\nu_i\geqslant\lambda_i$ for all i. Hence interchanging the roles of λ and μ

$$\{\nu/\mu\}=0 \qquad \text{if } \mu_i>\nu_i \text{ for any } i. \tag{4.4}$$

For example

$$\{21^3\}/\{2^2\}=0.$$

The relations (4.2) and (4.3) define the two fundamental operations on S-functions required here: multiplication and division, signified by and / respectively. Of course the trivial operations of addition and subtraction are signified by + and − respectively. The algebra of S-functions is such that

$$\begin{aligned}
&\{\lambda\cdot\mu\}=\{\mu\cdot\lambda\}\\
&\{\lambda\cdot\mu\cdot\nu\}=\{(\lambda\cdot\mu)\cdot\nu\}=\{\lambda\cdot(\mu\cdot\nu)\}\\
&\{\lambda\cdot(\mu+\nu)\}=\{\lambda\cdot\mu\}+\{\lambda\cdot\nu\}\\
&\{\lambda/\mu\cdot\nu\}=\{\lambda/(\mu\cdot\nu)\}=\{(\lambda/\mu)/\nu\}\\
&\{\lambda/\mu+\nu\}=\{\lambda/(\mu+\nu)\}=\{\lambda/\mu\}+\{\lambda/\nu\}.
\end{aligned}$$

S-function series play a central role in practical calculations and the following are particularly useful (King 1975b, King *et al* 1981):

$$A=\sum_{\alpha}(-1)^{a/2}\{\alpha\}\qquad B=\sum_{\beta}\{\beta\}\tag{4.5a}$$

$$C=\sum_{\gamma}(-1)^{c/2}\{\gamma\}\qquad D=\sum_{\delta}\{\delta\}\tag{4.5b}$$

$$E=\sum_{\varepsilon}(-1)^{(e+r)/2}\{\varepsilon\}\qquad F=\sum_{\zeta}\{\zeta\}\tag{4.5c}$$

$$G=\sum_{\varepsilon}(-1)^{(e-r)/2}\{\varepsilon\}\qquad H=\sum_{\zeta}(-1)^{z}\{\zeta\}\tag{4.5d}$$

$$L=\sum_{m}(-1)^{m}\{1^{m}\}\qquad M=\sum_{m}\{m\}\tag{4.5e}$$

$$P=\sum_{m}(-1)^{m}\{m\}\qquad Q=\sum_{m}\{1^{m}\}\tag{4.5f}$$

$$V=\sum_{\omega}(-1)^{q}\{\tilde{\omega}\}\qquad W=\sum_{\omega}(-1)^{q}\{\omega\}\tag{4.5g}$$

$$X=\sum_{\omega}\{\tilde{\omega}\}\qquad Y=\sum_{\omega}\{\omega\}\tag{4.5h}$$

where α, ε and γ are partitions of a, e and c respectively which take the form

$$\begin{aligned}
&\alpha=\begin{pmatrix}a_1 & a_2 & \ldots & a_r\\ a_1+1 & a_2+1 & \ldots & a_r+1\end{pmatrix}\qquad \varepsilon=\begin{pmatrix}a_1 & a_2 & \ldots & a_r\\ a_1 & a_2 & \ldots & a_r\end{pmatrix}\\
&\gamma=\begin{pmatrix}a_1 & a_2 & \ldots & a_r\\ a_1-1 & a_2-1 & \ldots & a_r-1\end{pmatrix}
\end{aligned}\tag{4.6}$$

where r is the Frobenius rank of the partition; δ is a partition all of whose parts are even; β is conjugate to δ; ζ is any partition; m is a one-part partition and ω is a partition of an even number into at most two parts, the second of which is q. The first few terms of each of these infinite S-function series are displayed in table 6.

The identification of these series by means of capital letters gives a compact notation and leads to a concise form of symbolic manipulation. This is typified by the definitions

$$\{\lambda\cdot A\}=\sum_{\alpha}(-1)^{a/2}\{\lambda\cdot\alpha\}$$

Table 6. *S*-function series.

$$
\begin{aligned}
A &= \{0\}-\{1^2\}+\{21^2\}-\{31^3\}-\{2^3\}+\{32^21\}-\{3^22^2\}+\{41^4\}-\{42^21^2\}+\{3^4\}+\ldots\\
B &= \{0\}+\{1^2\}+\{2^2\}+\{1^4\}+\{2^21^2\}+\{3^2\}+\{1^6\}+\{2^4\}+\{3^21^2\}+\{2^21^4\}+\{1^8\}+\ldots\\
C &= \{0\}-\{2\}+\{31\}-\{41^2\}-\{3^2\}+\{431\}+\{51^3\}-\{61^4\}-\{531^2\}-\{4^22\}+\{4^3\}+\ldots\\
D &= \{0\}+\{2\}+\{2^2\}+\{4\}+\{42\}+\{2^3\}+\{6\}+\{4^2\}+\{42^2\}+\{62\}+\{8\}+\ldots\\
E &= \{0\}-\{1\}+\{21\}-\{2^2\}-\{31^2\}+\{321\}+\{41^3\}-\{3^22\}+\{3^3\}-\{421^2\}+\ldots\\
F &= \{0\}+\{1\}+\{1^2\}+\{2\}+\{3\}+\{21\}+\{1^3\}+\{4\}+\{31\}+\{2^2\}+\{21^2\}+\{1^4\}+\ldots\\
G &= \{0\}+\{1\}-\{21\}-\{2^2\}+\{31^2\}+\{321\}-\{41^3\}-\{3^32\}-\{3^3\}-\{421^2\}+\ldots\\
H &= \{0\}-\{1\}+\{1^2\}+\{2\}-\{3\}-\{21\}-\{1^3\}+\{4\}+\{31\}+\{2^2\}+\{21^2\}+\{1^4\}+\ldots\\
L &= \{0\}-\{1\}+\{1^2\}-\{1^3\}+\{1^4\}-\{1^5\}+\{1^6\}-\{1^7\}+\{1^8\}-\{1^9\}+\{1^{10}\}+\ldots\\
M &= \{0\}+\{1\}+\{2\}+\{3\}+\{4\}+\{5\}+\{6\}+\{7\}+\{8\}+\{9\}+\{10\}+\ldots\\
P &= \{0\}-\{1\}+\{2\}-\{3\}+\{4\}-\{5\}+\{6\}-\{7\}+\{8\}-\{9\}+\{10\}+\ldots\\
Q &= \{0\}+\{1\}+\{1^2\}+\{1^3\}+\{1^4\}+\{1^5\}+\{1^6\}+\{1^7\}+\{1^8\}+\{1^9\}+\{1^{10}\}+\ldots\\
V &= \{0\}+\{1^2\}-\{2\}+\{1^4\}-\{21^2\}+\{2^2\}+\{1^6\}-\{21^4\}+\{2^21^2\}-\{2^3\}+\ldots\\
W &= \{0\}+\{2\}-\{1^2\}+\{4\}-\{31\}+\{2^2\}+\{6\}-\{51\}+\{42\}-\{3^2\}+\ldots\\
X &= \{0\}+\{1^2\}+\{2\}+\{1^4\}+\{21^2\}+\{2^2\}+\{1^6\}+\{21^4\}+\{2^21^2\}+\{2^3\}\ldots\\
Y &= \{0\}+\{2\}+\{1^2\}+\{4\}+\{31\}+\{2^2\}+\{6\}+\{51\}+\{42\}+\{3^2\}+\ldots
\end{aligned}
$$

$$\{\lambda/A\} = \sum_\alpha (-1)^{a/2}\{\lambda/\alpha\}$$

$$\{\lambda/\mu\cdot A\} = \sum_\alpha (-1)^{a/2}\{\lambda/\mu\cdot\alpha\}$$

so that for example

$$
\begin{aligned}
\{1^2\cdot A\} &= \{1^2\}-\{1^2\cdot 1^2\}+\{1^2\cdot 21^2\}-\ldots\\
&= \{1^2\}-\{2^2\}-\{21^2\}-\{1^4\}+\{321\}+\{31^3\}+\{2^3\}+\{2^21^2\}+\{21^4\}-\ldots\\
\{2^21/A\} &= \{2^21\}-\{2^21/1^2\}+\{2^21/21^2\}-\{2^21/31^3\}-\ldots\\
&= \{2^21\}-\{21\}-\{1^3\}+\{1\}\\
\{2^21/1^2\cdot A\} &= \{2^21/1^2\}-\{2^21/2^2\}-\{2^21/21^2\}-\ldots\\
&= \{21/A\}+\{1^3/A\} = \{21\}+\{1^3\}+2\{1\}.
\end{aligned}
$$

It should be noted that the series (4.5) have been arranged in mutually inverse pairs so that

$$AB = CD = EF = GH = LM = PQ = VW = 1 = \{0\} \tag{4.7}$$

where, for example,

$$AB = \sum_{\alpha,\beta} (-1)^{a/2}\{\alpha\cdot\beta\}.$$

The following identities are also satisfied:

$$AL = CP = E \qquad BM = DQ = F \tag{4.8a}$$

$$CM = AQ = G \qquad DL = BP = H \tag{4.8b}$$

$$MP = AD = W \qquad LQ = BC = V. \tag{4.8c}$$

It is also convenient to introduce $Q_\pm = (Q \pm L)/2$ and $X_\pm = (X \pm V)/2$ so that

$$Q_+ = \sum_m \{1^{2m}\} \qquad Q_- = \sum_m \{1^{2m+1}\} \tag{4.9a}$$

$$X_+ = \sum_{m,p} \{2^{2m-2p}1^{2p}\} \qquad X_- = \sum_{m,p} \{2^{2m-2p+1}1^{2p}\}. \tag{4.9b}$$

Each of these series, like B, D, M and Q, contains only positive contributions:

$$Q_+ = \{0\} + \{1^2\} + \{1^4\} + \{1^6\} + \{1^8\} + \{1^{10}\} + \ldots$$

$$Q_- = \{1\} + \{1^3\} + \{1^5\} + \{1^7\} + \{1^9\} + \{1^{11}\} + \ldots$$

$$X_+ = \{0\} + \{1^2\} + \{2^2\} + \{1^4\} + \{2^2 1^2\} + \{1^6\} + \ldots$$

$$X_- = \{2\} + \{21^2\} + \{2^3\} + \{21^4\} + \ldots.$$

5. Kronecker products for U_n, SU_n, Sp_{2k}, O_n and SO_{2k+1}

As pointed out in § 4, the Kronecker product of two covariant tensor irreps $\{\lambda\}$ and $\{\mu\}$ of U_n is determined by the S-function product rule (4.2)

$$U_n \qquad \{\lambda\} \times \{\mu\} = \{\lambda \cdot \mu\} = \sum_\nu m^\nu_{\lambda\mu}\{\nu\}. \tag{5.1}$$

The rule appropriate to Kronecker products of mixed tensors is more complicated, due to the additional constraint that the mixed tensors associated with irreps must be traceless under the contraction of any contravariant index with a covariant index. The necessary symmetrisation and extraction of traces leads to the compact formula (Abramsky and King 1970, King 1971)

$$U_n \qquad \{\bar{\mu}; \lambda\} \times \{\bar{\rho}; \nu\} = \sum_{\sigma,\tau} \{\overline{(\mu/\sigma) \cdot (\rho/\tau)}; (\lambda/\tau) \cdot (\nu/\sigma)\}. \tag{5.2}$$

This result, like the special case (5.1), is n-independent, although some of the terms in the final expression may not be standard for a particular value of n. Such terms need to be modified using the rule for irreps of U_n given in table 3.

In order to avoid the apparent complexities of (5.2) it is possible to make use of the special irrep ε to associate a covariant tensor irrep with each mixed tensor irrep in accordance with the identity

$$\{\bar{\mu}; \lambda\} = \bar{\varepsilon}^{\mu_1}\{\mu_1 + \lambda_1, \mu_1 + \lambda_2, \ldots, \mu_1 - \mu_3, \mu_1 - \mu_2, 0\}, \tag{5.3}$$

which is the generalisation of (3.1c) from SU_n to U_n.

By way of example (5.2) yields the general result

$$U_n \qquad \{\bar{1}; 1\} \times \{\bar{1}; 1\} = \{\bar{2}; 2\} + \{\bar{2}; 1^2\} + \{\bar{1}^2; 2\} + \{\bar{1}^2; 1^2\} + 2\{\bar{1}; 1\} + \{0\}.$$

However, the term $\{\bar{1}^2; 1^2\}$ is not U_3-standard. Under the modification rule for U_3

in table 3 it is found to be zero and thus does not contribute to the result in the case $n=3$:

$$U_3 \qquad \{\bar{1};1\}\times\{\bar{1};1\}=\{\bar{2};2\}+\{\bar{2};1^2\}+\{\bar{1}^2;2\}+2\{\bar{1};1\}+\{0\}.$$

The alternative method is to note that for U_3, (5.3) gives $\{\bar{1};1\}=\bar{\varepsilon}\{21\}$. Hence using (5.1), the modification rule for U_3 and (5.3)

$$\begin{aligned} U_3 \qquad & \{\bar{1};1\}\times\{\bar{1};1\} \\ & =\bar{\varepsilon}^2\{21\}\times\{21\} \\ & =\bar{\varepsilon}^2(\{42\}+\{41^2\}+\{3^2\}+2\{321\}+\{31^3\}+\{2^3\}+\{2^21^2\}) \\ & =\bar{\varepsilon}^2(\{42\}+\varepsilon\{3\}+\{3^2\}+2\varepsilon\{21\}+\varepsilon^2\{0\}) \\ & =\{\bar{2};2\}+\{\bar{1}^2;2\}+\{\bar{2};1^2\}+2\{\bar{1};1\}+\{0\}. \end{aligned}$$

However, this result is only valid for U_3 and the method used is generally less efficient than the use of (5.2). Nonetheless the use of ε, as in (5.3), does have certain advantages as will be demonstrated in dealing with Kronecker products of irreps of SO_{2k}.

In the case of SU_n Kronecker products may be evaluated through the use of (5.1) and the simple application of the modification rules (3.1*a*) and (3.2). For example

$$SU_n \qquad \{21\}\times\{1^2\}=\{32\}+\{31^2\}+\{2^21\}+\{21^3\}$$

for $n\geqslant 5$, whilst for $n=3$

$$SU_3 \qquad \{21\}\times\{1^2\}=\{32\}+\{2\}+\{1^2\}.$$

Under the restriction from U_{2k} to Sp_{2k} the irrep $\{\lambda\}$ is, in general, no longer irreducible. The reduction is well known. In terms of the *S*-function series of (4.5):

$$U_{2k}\downarrow Sp_{2k} \qquad \{\lambda\}\downarrow\langle\lambda/B\rangle. \tag{5.4}$$

This decomposition has a converse obtained using the series A, inverse to B, (King 1975b)

$$Sp_{2k}\uparrow U_{2k} \qquad \langle\lambda\rangle\uparrow\{\lambda/A\}. \tag{5.5}$$

Hence Kronecker products of Sp_{2k} may be evaluated using the formula

$$Sp_{2k} \qquad \langle\lambda\rangle\times\langle\mu\rangle=\langle((\lambda/A)\cdot(\mu/A))/B\rangle. \tag{5.6}$$

This result is very complex, involving two infinite series and leading to both positive and negative terms. A great simplification has been provided by Newell (1951) and Littlewood (1958) who have shown that

$$Sp_{2k} \qquad \langle\lambda\rangle\times\langle\mu\rangle=\sum_{\zeta}\langle(\lambda/\zeta)\cdot(\mu/\zeta)\rangle. \tag{5.7}$$

This has a simple interpretation in terms of contractions with the metric tensor of Sp_{2k} followed by symmetrisation. It has the great merit of involving only positive terms. The result is k-independent but may require the use of the modification rule for Sp_{2k} given in table 3, for any specific value of k.

The use of (5.7) is exemplified by the products†

$$\begin{aligned}
\mathrm{Sp}_{2k} \qquad \langle 1\rangle\times\langle 21\rangle &= \langle 1\cdot 21\rangle+\langle 0\cdot(2+1^2)\rangle\\
&= \langle 31\rangle+\langle 2^2\rangle+\langle 21^2\rangle+\langle 2\rangle+\langle 1^2\rangle \qquad \text{for } k\geqslant 3\\
\langle 1^2\rangle\times\langle 21\rangle &= \langle 1^2\cdot 21\rangle+\langle 1\cdot(2+1^2)\rangle+\langle 0\cdot 1\rangle\\
&= \langle 32\rangle+\langle 31^2\rangle+\langle 2^21\rangle+\langle 21^3\rangle+\langle 3\rangle\\
&\quad +2\langle 21\rangle+\langle 1^3\rangle+\langle 1\rangle \qquad \text{for } k\geqslant 4\\
\langle 21\rangle\times\langle 21\rangle &= \langle 21\cdot 21\rangle+\langle(2+1^2)\cdot(2+1^2)\rangle+2\langle 1\cdot 1\rangle+\langle 0\cdot 0\rangle\\
&= \langle 42\rangle+\langle 41^2\rangle+\langle 3^2\rangle+2\langle 321\rangle+\langle 31^3\rangle+\langle 2^3\rangle\\
&\quad +\langle 2^21^2\rangle+\langle 4\rangle+3\langle 31\rangle+2\langle 2^2\rangle+3\langle 21^2\rangle+\langle 1^4\rangle\\
&\quad +2\langle 2\rangle+2\langle 1^2\rangle+\langle 0\rangle \qquad \text{for } k\geqslant 4.
\end{aligned}$$

For values of k smaller than those indicated, non-standard irrep labels arise and these must be modified using the appropriate formula of table 3. For example‡, in the case $k=2$

$$\mathrm{Sp}_4 \qquad \langle 1^2\rangle\times\langle 21\rangle=\langle 32\rangle+\langle 3\rangle+\langle 21\rangle+\langle 1\rangle.$$

As explained in § 2 the group O_n has two different types of irrep: tensor and spinor. There are therefore three different types of product to consider: tensor–tensor, tensor–spinor and spinor–spinor. The tensor–tensor products can be evaluated in the same way as the products of irreps of Sp_{2k}. The analogues of (5.4)–(5.7) are (King 1975b)

$$\mathrm{U}_n \downarrow \mathrm{O}_n \qquad \{\lambda\}\downarrow[\lambda/D] \tag{5.8}$$

$$\mathrm{O}_n \uparrow \mathrm{U}_n \qquad [\lambda]\uparrow\{\lambda/C\} \tag{5.9}$$

$$\mathrm{O}_n \qquad [\lambda]\times[\mu]=[((\lambda/C)\cdot(\mu/C))/D]. \tag{5.10}$$

This can again be simplified (Newell 1951, Littlewood 1958) to give§

$$\mathrm{O}_n \qquad [\lambda]\times[\mu]=\sum_{\zeta}[(\lambda/\zeta)\cdot(\mu/\zeta)]. \tag{5.11}$$

This result differs from (5.7) only in the application of the modification rules for O_n given in table 3.

† The application of Rules 1–2f of Fischler for Sp_{2k} gives incorrect results, for example, for each of the products $\langle 1\rangle\times\langle 21\rangle$ and $\langle 21\rangle\times\langle 21\rangle$. In the first case they give no term $\langle 2\rangle$, and in the second only one term $\langle 2^2\rangle$. The origin of this discrepancy lies in the fact that the single lattice permutation rule embodied in Rule 2, involving both added and cancelled boxes, does not properly take the place of the three distinct lattice permutation rules implicit in (5.7). For example, Rule 2 excludes the term $\langle 2\rangle$ in $\langle 1\rangle\times\langle 21\rangle$ which could be obtained via $a\to 1$, 1 cancels, $a\to 1, 1$, $b\to 2,1$. The only other way of obtaining a term $\langle 2\rangle$ would be via $a\to 1$, 1 cancels, $a\to 2,1$, $b\to 1, 1$. This also violates Rule 2 in the absence of any generalisation of Rule 2f to allow for the anticipation of future additions when cancelling.

‡ The application of Fischler's Rules 1–2f to the product $\langle 1^2\rangle\times\langle 21\rangle$ yields the correct result for Sp_{2k} provided that $k\geqslant 3$. However, this is no longer true in the case $k=2$, since Rule 2d then leads to the exclusion of both terms $\langle 21\rangle$. This is because Rule 2d does not give properly all the modifications appropriate to Sp_{2k} for small values of k.

§ The fact that (5.11) takes the same form as (5.7) corresponds to the fact that Fischler's Rules 2–2c and 2f apply to SO_{2k+1} and SO_{2k} as well as to Sp_{2k}. Unfortunately this means that the criticism of these Rules in the footnote † (above) applies not only to the case of Sp_{2k} but also to the cases of SO_{2k+1} and SO_{2k}.

A general method for evaluating tensor–spinor and spinor–spinor products of irreps of O_n has been developed (Butler and Wybourne 1969) and much simplified (King 1975a). Crucial to these developments is the result for the Kronecker square of the basic spin irrep Δ. This was first given by Brauer and Weyl (1935), and can be written in the form

$$O_{2k} \qquad \Delta^2=[Q]=[Q^*]=\sum_{m=0}^{k-1}([1^m]+[1^m]^*)+[1^k] \tag{5.12a}$$

$$O_{2k+1} \qquad \Delta^2=\tfrac{1}{2}[Q^*]=\sum_{m=0}^{k}[1^m]^{(*)^m}. \tag{5.12b}$$

The series Q^* and its inverse P^* are defined by

$$Q^*=\sum_m \{1^m\}^{(*)^m} \qquad P^*=\sum_m (-1)^m\{m\}^{(*)^m} \tag{5.13}$$

and use has been made of the modification rules for O_{2k} and O_{2k+1} to give the final expressions in (5.12).

To evaluate products involving the spinor irreps $[\Delta;\lambda]$ it is necessary to know how such an irrep can be expressed as a product of the basic spin irrep Δ and a sum of tensor irreps. The appropriate relations and their inverses are given by (King 1975a, King *et al* 1981)

$$O_{2k} \qquad [\Delta;\lambda]=\Delta\times[\lambda/P]=\sum_m(-1)^m\Delta\times[\lambda/m] \tag{5.14a}$$

$$\Delta\times[\lambda]=[\Delta;\lambda/Q]=\sum_m[\Delta;\lambda/1^m] \tag{5.14b}$$

$$O_{2k+1} \qquad [\Delta;\lambda]=\Delta\times[\lambda/P^*]=\sum_m\Delta\times([\lambda/2m]-[\lambda/(2m+1)]^*) \tag{5.14c}$$

$$\Delta\times[\lambda]=[\Delta;\lambda/Q^*]=\sum_m([\Delta;\lambda/1^{2m}]+[\Delta;\lambda/1^{2m+1}]^*). \tag{5.14d}$$

Then by simple manipulations involving the identity

$$\{\lambda\cdot\mu\}/Z=\{\lambda/Z\}\cdot\{\mu/Z\} \tag{5.15}$$

which is valid for the S-function series $Z=L$, M, P, Q, V and W, the following Kronecker product formulae may be derived (King 1975a, King *et al* 1981):

$$O_{2k} \qquad [\Delta;\lambda]\times[\mu]=\sum_\zeta[\Delta;(\lambda/\zeta)\cdot(\mu/\zeta Q)] \tag{5.16a}$$

$$[\Delta;\lambda]\times[\Delta;\mu]=\sum_\zeta[(\lambda/\zeta)\cdot(\mu/\zeta)\cdot Q] \tag{5.16b}$$

$$O_{2k+1} \qquad [\Delta;\lambda]\times[\mu]=\sum_\zeta[\Delta;(\lambda/\zeta)\cdot(\mu/\zeta Q^*)] \tag{5.16c}$$

$$[\Delta;\lambda]\times[\Delta;\mu]=\sum_\zeta\tfrac{1}{2}[(\lambda/\zeta)\cdot(\mu/\zeta)\cdot Q^*]. \tag{5.16d}$$

These together with (5.11) complete the required set of rules for O_n. Moreover (5.11), (5.16*a*) and (5.16*c*) each contain only a finite number of terms. In all cases the summation over ζ is finite since ζ only enters as a divisor of λ and μ. By virtue of (4.4) it follows that only a finite number of non-vanishing terms may arise. Similarly in (5.16*a*) and (5.16*c*) only a finite number of terms in the series Q and Q^* give rise to non-vanishing contributions. Unfortunately (5.16*b*) and (5.16*d*), like (5.12), contain

S-function series bounded only by the modification rules. In practice this creates a not insignificant overcounting problem as the O_n modification rules do not provide a very efficient cut-off. The situation may be improved (King *et al* 1981) by performing various manipulations on the formulae (5.16*b*) and (5.16*d*) which allow modification to take place at an intermediate stage. In this way fewer redundant terms are produced but at the cost of simplicity in the statement of the general result.

For the group SO_n the situation regarding products is even more difficult. The problem lies in the reducibility of the self-associated irreps of O_n under restriction to SO_n. There is no difficulty for O_{2k+1} since all the irreps remain irreducible on restriction to SO_{2k+1}. The equations (5.11), (5.16*c*) and (5.16*d*) remain valid but now the * is irrelevant for SO_{2k+1} and may be deleted from the last two equations.

Hence for example from (5.11)

$$\begin{aligned} SO_7 \qquad [1^3]\times[1^2] &= [1^3\cdot 1^2]+[1^2\cdot 1]+[1\cdot 0] \\ &= [2^21]+[21^3]+[1^5]+[21]+[1^3]+[1] \\ &= [2^21]+[21^2]+[1^2]+[21]+[1^3]+[1] \end{aligned}$$

where use has been made of the modifications†

$$SO_7 \qquad [21^3]=[21^2] \qquad \text{and} \qquad [1^5]=[1^2].$$

Similarly from (5.16*c*)‡

$$\begin{aligned} SO_5 \qquad [1^2]\times[\Delta;1^2] &= [\Delta;1^2\cdot 1^2/Q]+[\Delta;1\cdot 1/Q]+[\Delta;0\cdot 0/Q] \\ &= [\Delta;1^2\cdot(1^2+1+0)]+[\Delta;1\cdot(1+0)]+[\Delta;0] \\ &= [\Delta;2^2+21^2+1^4+21+1^3+1^2+2+1^2+1+0] \\ &= [\Delta;2^2]+[\Delta;21]+[\Delta;2]+[\Delta;1^2]+[\Delta;1]+[\Delta] \end{aligned}$$

since from the modification rules of table 3

$$SO_5 \qquad [\Delta;21^2]=[\Delta;1^3]=0 \qquad \text{and} \qquad [\Delta;1^4]=-[\Delta;1^2].$$

Finally from (5.16*d*)§

$$\begin{aligned} SO_5 \qquad & [\Delta;1^2]\times[\Delta;1^2] \\ &= \tfrac{1}{2}[(1^2\cdot 1^2+1\cdot 1+0\cdot 0)\cdot Q] \\ &= \tfrac{1}{2}[(2^2+21^2+1^4+2+1^2+0)\cdot(0+1+1^2+1^3+\ldots)] \end{aligned}$$

† These modification rules are not adequately represented by Fischler's Rules 1d, 3 and 3a as can be seen by considering the product $[1^3]\times[1^2]$ in SO_7. Unless Rule 3a is extended to cases for which R_2 does not necessarily contain a column of N boxes, this Rule does not apply and no term $[1^2]$ is obtained. Instead the application of Rule 1d leads to an unwanted second term $[1^3]$ via $a\to 1,3$ merge, $b\to 1,3$ merge. Altering the rules leads only to difficulties with other products.

‡ Instead of (5.16*c*) Fischler's Rule 3c involves evaluating $[\Delta;([\lambda]\times[\mu])/Q]$ and then reducing by 1 all multiplicities greater than 1. The product $[\lambda]\times[\mu]$ is to be evaluated in SO_{2k+1}. This procedure gives the correct result in some cases but fails, for example, in the case of the SO_5 product $[1^2]\times[\Delta;1^2]$. Amongst other errors Rule 3c leads to a final multiplicity of 2 for the term Δ in this product. This failure is once more attributable to the fact that Fischler's Rules do not reproduce correctly all the necessary modifications.

§ In contrast to (5.16*d*) Fischler's Rule 3d involves evaluating $[\{[\lambda]\times[\mu]\}\cdot\{Q\}]$ with no subsequent reduction of multiplicities. Once more the product $[\lambda]\times[\mu]$ is to be evaluated in SO_{2k+1} and the brackets $\{\ldots\}$ have been included to indicate that the final multiplication by Q is to be carried out as if for U_k. This attractively simple rule fails, for example, in the case of the SO_5 product $[\Delta;1^2]\times[\Delta;1^2]$ since, amongst other errors, it yields a multiplicity 3 for the term $[2^2]$. The modification rules are not adequately covered by this oversimplified Rule 3d.

$$=\tfrac{1}{2}[(2^2+21^2+1^4+2+1^2+0+32+2^21+31^2+2^21$$
$$+21^3+21^3+1^5+3+21+21+1^3+1+3^2+\ldots)]$$
$$=\tfrac{1}{2}[(2^2+21+1+2+1^2+0+32+2^2+31+2^2+2$$
$$+2+0+3+21+21+1^2+1+3^2+\ldots)]$$
$$=[3^2]+[32]+[31]+[3]+[2^2]+[21]+[2]+[1^2]+[1]+[0],$$

where the modification rules appropriate to SO_5 have been used to standardise the irrep labels and to terminate the infinite series.

The case of SO_{2k} is rather different since each self-associate irrep of O_{2k} yields two inequivalent irreps of SO_{2k} in accordance with the branching rules

$$O_{2k}\downarrow SO_{2k}\qquad [\lambda]\downarrow[\lambda]=[\lambda]_++[\lambda]_- \qquad \text{if } p=k \tag{5.17a}$$
$$[\Delta;\lambda]\downarrow[\Delta;\lambda]=[\Delta;\lambda]_++[\Delta;\lambda]_- \qquad \text{for } p\leqslant k \tag{5.17b}$$

where the first of these can equally well be written in the form

$$[\Box;\lambda]\downarrow[\Box;\lambda]=[\Box;\lambda]_++[\Box;\lambda]_- \qquad \text{for } p\leqslant k. \tag{5.17c}$$

Considerable progress has been made through the use of difference characters (Murnaghan 1938, pp 290, 315; Littlewood 1940, pp 246, 259) for both tensor and spinor representations. These difference characters are defined by

$$SO_{2k}\qquad [\lambda]''=[\lambda]_+-[\lambda]_- \tag{5.18a}$$
$$[\Delta;\lambda]''=[\Delta;\lambda]_+-[\Delta;\lambda]_- \tag{5.18b}$$
$$[\Box;\lambda]''=[\Box;\lambda]_+-[\Box;\lambda]_- \tag{5.18c}$$

so that

$$SO_{2k}\qquad [\lambda]_\pm=\tfrac{1}{2}([\lambda]\pm[\lambda]'') \tag{5.19a}$$
$$[\Delta;\lambda]_\pm=\tfrac{1}{2}([\Delta;\lambda]\pm[\Delta;\lambda]'') \tag{5.19b}$$
$$[\Box;\lambda]_\pm=\tfrac{1}{2}([\Box;\lambda]\pm[\Box;\lambda]''). \tag{5.19c}$$

Many results have been produced for Kronecker products of irreps of SO_{2k} using difference characters in conjunction with S-function manipulations (Butler and Wybourne 1969, King *et al* 1981). However, these are not entirely satisfactory in that the formulae obtained are lengthy and cumbersome. For this reason we content ourselves, at this stage, with the remark that difference characters are used in the appendix to derive branching rules appropriate to the restriction from SO_{2k} to U_k. These are employed later in the derivation of Kronecker product formulae for irreps of SO_{2k} by means of a new technique described in § 6.

6. The branching rule method

Given an irrep λ_G of a group G, the restriction of the group elements to those of a subgroup H gives a representation which is, in general, reducible. The branching rule problem is that of determining the irreps σ_H of H which appear as constituents of λ_G on restriction from G to H, along with their branching multiplicities $B^{\sigma_H}_{\lambda_G}$. The corresponding branching rule takes the form

$$G \downarrow H \qquad \lambda_G \downarrow \sum_{\sigma_H} B^{\sigma_H}_{\lambda_G} \sigma_H. \tag{6.1}$$

In many instances this branching rule has an inverse

$$H \uparrow G \qquad \sigma_H \uparrow \sum_{\lambda_G} A^{\lambda_G}_{\sigma_H} \lambda_G \tag{6.2}$$

and the solutions to the problem of determining the coefficients in (6.1) and (6.2) are of great use in evaluating Kronecker products in G from a knowledge of those in H and indeed *vice versa*.

To be explicit, if the Kronecker products of irreps of G and H are such that

$$G \qquad \lambda_G \times \mu_G = \sum_{\nu_G} K^{\nu_G}_{\lambda_G \mu_G} \nu_G \tag{6.3}$$

and

$$H \qquad \sigma_H \times \tau_H = \sum_{\rho_H} K^{\rho_H}_{\sigma_H \tau_H} \rho_H \tag{6.4}$$

then the Kronecker product multiplicities of G are given by

$$K^{\nu_G}_{\lambda_G \mu_G} = \sum_{\sigma_H, \tau_H, \rho_H} B^{\sigma_H}_{\lambda_G} B^{\tau_H}_{\mu_G} K^{\rho_H}_{\sigma_H \tau_H} A^{\nu_G}_{\rho_H} \tag{6.5}$$

with a similar formula expressing the Kronecker product multiplicities of H in terms of those of G.

This technique has already been exploited in § 5 in the derivation of (5.6) and (5.10) for the Kronecker product of tensor irreps of Sp_{2k} and O_n respectively. Similar methods using subgroups have also been used (Wybourne and Bowick 1977) to evaluate Kronecker products of irreps of the exceptional groups.

In the special case for which H is the maximal toroidal subgroup, $\mathrm{T} = \mathrm{U}_1 \times \mathrm{U}_1 \times \ldots \times \mathrm{U}_1$ of G, (6.1) takes the form

$$G \downarrow \mathrm{T} \qquad \boldsymbol{\lambda}_G \downarrow \sum_{\boldsymbol{\sigma}_\mathrm{T}} M^{\boldsymbol{\sigma}_\mathrm{T}}_{\lambda_G} \boldsymbol{\sigma}_\mathrm{T} \tag{6.6}$$

where $\boldsymbol{\sigma}_\mathrm{T}$ is a one-dimensional irrep $\{\sigma_1\} \times \{\sigma_2\} \times \ldots \times \{\sigma_k\}$ of T and the coefficients appearing in this expression are nothing other than the weight multiplicities of the irrep $\boldsymbol{\lambda}_G$. The use of (6.5) in this case provides a weight space method of evaluating Kronecker products which may in fact be used to derive the product rule (5.1) appropriate to Schur functions and tensor irreps of U_n.

Such weight space techniques can be simplified thanks to the work of Racah (1964) and Speiser (1964) who derived the result

$$G \qquad \boldsymbol{\lambda}_G \times \boldsymbol{\mu}_G = \sum_{\boldsymbol{\sigma}_\mathrm{T}} M^{\boldsymbol{\sigma}_\mathrm{T}}_{\lambda_G} (\boldsymbol{\mu}_G + \boldsymbol{\sigma}_\mathrm{T})_G \tag{6.7}$$

where $(\boldsymbol{\mu}_G + \boldsymbol{\sigma}_\mathrm{T})_G$ denotes the G-standard label $\boldsymbol{\nu}_G$ for the irrep of G equivalent to the irrep labelled by $\boldsymbol{\mu}_G + \boldsymbol{\sigma}_\mathrm{T}$. To exploit this formula it is thus necessary to make use of the modification rules of § 3, table 4. The only other information required is the set of weights $\boldsymbol{\sigma}_\mathrm{T}$, and their multiplicities, for just one of the irreps $\boldsymbol{\lambda}_G$ of the product $\boldsymbol{\lambda}_G \times \boldsymbol{\mu}_G$. Alternatively the use of modification rules may, in this case, be avoided by means of a technique due to Englefield (1981) involving the dimensions of irreps and the eigenvalues of a set of Casimir operators.

More recently a new technique for evaluating Kronecker products has been devised (King 1981). It is a generalisation of (6.7) with T replaced by a subgroup H of G which has the same rank as G and which is embedded naturally in G. The derivation leans heavily on the work of Racah (1964) and Speiser (1964) and yields the formula

$$G \qquad \boldsymbol{\lambda}_G \times \boldsymbol{\mu}_G = \sum_{\sigma_H, \rho_H} B_{\lambda_G}^{\sigma_H} K_{\sigma_H, \mu_G + \delta_G - \delta_H}^{\rho_H} (\boldsymbol{\rho}_H - \boldsymbol{\delta}_G + \boldsymbol{\delta}_H)_G \tag{6.8}$$

where $\boldsymbol{\delta}_G$ and $\boldsymbol{\delta}_H$ are equal to half the sums of the positive roots of G and H respectively (King and Al-Qubanchi 1981a).

To use this result it is necessary to identify a suitable subgroup H of G and to know four things about these groups: firstly the branching rule for the restriction from G to H in the case of the irrep $\boldsymbol{\lambda}_G$; secondly, $\boldsymbol{\delta}_G - \boldsymbol{\delta}_H$; thirdly, the Kronecker product multiplicities of H, since it is these multiplicities which appear in (6.8); and finally, the modification rules appropriate to G, in order to convert $(\boldsymbol{\rho}_H - \boldsymbol{\delta}_G + \boldsymbol{\delta}_H)$ to G-standard form.

For reasons of simplicity in evaluating Kronecker products in H it is convenient to choose H to be a unitary group of the same rank as G. Suitable group–subgroup combinations are given in table 7, together with $\boldsymbol{\delta}_G - \boldsymbol{\delta}_H$, which may be obtained from earlier work (King and Al-Qubanchi 1981a). The striking thing about this tabulation is that $\boldsymbol{\delta}_G - \boldsymbol{\delta}_H$ is null for SO_{2k} whilst for SO_{2k+1} and Sp_{2k} it is of the form $\varepsilon^{1/2}$ and ε respectively, where ε is the weight vector of the irrep $\{1^k\}$ of U_k used to relate associated irreps of this group as explained in § 2. Thus the addition and subtraction of $\boldsymbol{\delta}_G - \boldsymbol{\delta}_H$ in (6.8) is a null operation in all these cases. We may write for $G = SO_{2k+1}$, Sp_{2k}, SO_{2k} and $H = U_k$ the simpler formula

$$\boldsymbol{\lambda}_G \times \boldsymbol{\mu}_G = \sum_{\sigma_H, \rho_H} B_{\lambda_G}^{\sigma_H} K_{\sigma_H \mu_G}^{\rho_H} (\boldsymbol{\rho}_H)_G \tag{6.9}$$

where the final subscript, G, is used as in (6.7) and (6.8) to indicate the need to convert $\boldsymbol{\rho}_H$ to G-standard form. The implementation of this result is described in § 7.

Table 7. Relationship between G and a unitary subgroup H of the same rank.

G	H	$\boldsymbol{\delta}_G - \boldsymbol{\delta}_H$	$\boldsymbol{\delta}$
SO_{2k+1}	U_k	$(\frac{1}{2}\ \frac{1}{2} \ldots \frac{1}{2})$	$(0\ 0 \ldots 0)$
Sp_{2k}	U_k	$(1\ 1 \ldots 1)$	$(0\ 0 \ldots 0)$
SO_{2k}	U_k	$(0\ 0 \ldots 0)$	$(0\ 0 \ldots 0)$
G_2	SU_3	$(1\ 0)$	$(1,\ 0)$
F_4	U_4	$(\frac{5}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2})$	$(2, 000)$
E_6	$SU_2 \times SU_6$	$(10, 00000)$	$(10, 00000)$
E_7	SU_8	$(10, 000000)$	$(10, 000000)$
E_8	SU_9	$(21, 0000000)$	$(21, 0000000)$

In the case of the exceptional groups the situation is more complicated. For G_2, E_6, E_7 and E_8 it is to be noted that $\boldsymbol{\delta}_G - \boldsymbol{\delta}_H$ has a single non-vanishing component; namely the first component d which takes on the values 1, 10, 10 and 21 respectively. The most troublesome case seems to be that of F_4. However for $G = F_4$ and $H = U_4$ we can write $\boldsymbol{\delta}_G - \boldsymbol{\delta}_H = (\frac{5}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}) = (\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}) + (2000)$ where $(\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2})$ is nothing other than $\varepsilon^{1/2}$

once again. As in the case of the classical groups, the addition and subtraction of $\varepsilon^{1/2}$ may be omitted.

It follows that in all cases

$$\boldsymbol{\lambda}_G \times \boldsymbol{\mu}_G = \sum_{\boldsymbol{\sigma}_H, \boldsymbol{\rho}_H} B^{\boldsymbol{\sigma}_H}_{\boldsymbol{\lambda}_G} K^{\boldsymbol{\rho}_H}_{\boldsymbol{\sigma}_H, \boldsymbol{\mu}_G + \boldsymbol{\delta}} (\boldsymbol{\rho}_H - \boldsymbol{\delta})_G \tag{6.10}$$

where $\boldsymbol{\delta} = (d, 0, 0, \ldots, 0)$ is defined in table 7. The application of this formula to the exceptional groups is discussed in § 8.

7. Kronecker products for SO_{2k}, Sp_{2k} and SO_{2k+1}

In order to exploit the method of § 6, as summarised for the classical groups in (6.9), it is necessary to know the relevant branching rules. For $\mathrm{SO}_{2k} \downarrow \mathrm{U}_k$ these are given in the appendix, the required results being (A4), (A12) and (A18). It is then a simple matter to write down Kronecker product formulae for all irreps of SO_{2k}:

SO_{2k}

$$[\lambda] \times [\mu]_+ = \sum_\xi [\{\bar{\xi}; (\lambda/\xi B)\} \cdot \{\mu\}]_+ \tag{7.1a}$$

$$[\lambda] \times [\Delta; \mu]_+ = \sum_\xi [\varepsilon^{1/2} \cdot \{\bar{\xi}; (\lambda/\xi B)\} \cdot \{\mu\}]_+ \tag{7.1b}$$

$$[\lambda] \times [\Box; \mu]_+ = \sum_\xi [\varepsilon \cdot \{\bar{\xi}; (\lambda/\xi B)\} \cdot \{\mu\}]_+ \tag{7.1c}$$

$$[\Delta; \lambda]_\pm \times [\mu]_+ = \sum_\xi [\varepsilon^{1/2} \cdot \{\overline{\xi \cdot Q_\pm}; (\lambda/\xi B)\} \cdot \{\mu\}]_+ \tag{7.1d}$$

$$[\Delta; \lambda]_\pm \times [\Delta; \mu]_+ = \sum_\xi [\{\bar{\xi}; (\lambda/\xi B) \cdot Q_{\pm(-)^k}\} \cdot \{\mu\}]_+ \tag{7.1e}$$

$$[\Delta; \lambda]_\pm \times [\Box; \mu]_+ = \sum_\xi [\varepsilon^{1/2} \cdot \{\bar{\xi}; (\lambda/\xi B) \cdot Q_{\pm(-)^k}\} \cdot \{\mu\}]_+ \tag{7.1f}$$

$$[\Box; \lambda]_\pm \times [\mu]_+ = \sum_\xi [\varepsilon \cdot \{\overline{\xi \cdot X_\pm}; (\lambda/\xi B)\} \cdot \{\mu\}]_+ \tag{7.1g}$$

$$[\Box; \lambda]_\pm \times [\Delta; \mu]_+ = \sum_\xi [\varepsilon^{3/2} \cdot \{\overline{\xi \cdot X_\pm}; (\lambda/\xi B)\} \cdot \{\mu\}]_+ \tag{7.1h}$$

$$[\Box; \lambda]_\pm \times [\Box; \mu]_+ = \sum_\xi [\{\bar{\xi}; (\lambda/\xi B) \cdot X_{\pm(-)^k}\} \cdot \{\mu\}]_+ \tag{7.1i}$$

where $[\mu]_+ = [\mu]$ if the number of non-vanishing parts of μ is less than k.

All possible cases are covered by the use of the involutory outer automorphism, $\dagger$, of SO_{2k} whose action is defined by (2.5). Because of the commutative nature of Kronecker products there is some choice available in this set of formulae. They require for their implementation the evaluation and modification of products in U_k, with a final modification in SO_{2k}. Division by the series B for fixed λ only involves a finite number of terms as does the summation over ξ. The modification rules of U_k are used to terminate the series $Q_\pm$ and $X_\pm$, thus providing a more effective cut-off than do those of O_n in (5.16b) and (5.16d).

It is instructive to consider a few examples: firstly the use of (7.1a) in evaluating $[21] \times [21^3]_+$ for SO_8. The first step is to evaluate $\Sigma_\xi \{\bar{\xi}; (21/\xi B)\}$ which may be done from first principles or by using Butler's tabulation of S-function products and quotients

(Wybourne 1970). We obtain

$$\sum_{\xi}\{\bar{\xi};(21/\xi B)\}$$

$$=\{\bar{0};21/B\}+\{\bar{1};2/B\}+\{\bar{1};1^2/B\}+\{\bar{2};1/B\}+\{\overline{1^2};1/B\}+\{\overline{21};0/B\}$$
$$=\{\bar{0};21\}+\{\bar{0};1\}+\{\bar{1};2\}+\{\bar{1};1^2\}+\{\bar{1};0\}+\{\bar{2};1\}+\{\overline{1^2};1\}+\{\overline{21};0\}.$$

Each term must now be multiplied by $\{21^3\}=\varepsilon\{1\}$ in U_4, giving

$$\{421^2\}+\{41^2\}+\{3^21^2\}+\{32^21\}+2\{321\}+2\{31^3\}+\{\bar{1};31^2\}+\{31\}+\{2^3\}$$
$$+2\{2^21^2\}+\{\bar{1};2^21\}+\{2^2\}+3\{21^2\}+\{\bar{1};21\}+\{1^4\}+\{\bar{1};1^3\}+\{1^2\}.$$

Each term of this sum is U_4-standard but must be made SO_8-standard either using the appropriate modification rule for SO_8 given in table 5 or, more simply, using (2.4). Hence

$$\begin{aligned}\mathrm{SO}_8\qquad &[21]\times[21^3]_+\\ &=[421^2]_++[41^2]+[3^21^2]_++[32^21]_++2[321]+2[31^3]_+\\ &\quad+[31^3]_-+[31]+[2^3]+2[2^21^2]_++[2^21^2]_-+[2^2]\\ &\quad+3[21^2]+[1^4]_++[1^4]_-+[1^2].\end{aligned}\qquad(7.2)$$

The terms in $[21]\times[21^3]_-$ then follow by applying (2.5) to both sides.

As a second example consider $[\Delta;1]_-\times[2^21]_+$ in SO_6 evaluated by means of (7.1*d*) with $k=3$. Noting (4.9*a*), we first calculate

$$\sum_{\xi,m}\{\overline{\xi\cdot 1^{2m+1}};1/\xi B\}$$

$$=\sum_{\xi,m}\{\overline{\xi\cdot 1^{2m+1}};1/\xi\}=\sum_{m}\{\overline{1^{2m+1}};1\}+\sum_{m}\{\overline{1^{2m+1}\cdot 1};0\}$$
$$=\{\bar{1};1\}+\{\overline{1^2};0\}+\{\bar{2};0\}+\{\overline{21^2};0\}$$

where U_3 modification has cut off the series. This sum must now be multiplied by $\{2^21\}=\varepsilon\{1^2\}$ in U_3, giving

$$\{32\}+\{31^2\}+\{2^21\}+\{\bar{1};2^2\}+2\{21\}+\{1^3\}+\{\bar{1};1^2\}+\{1\}.$$

Interpreting the factor $\varepsilon^{1/2}$ from (7.1*d*) as Δ and subsequently modifying the terms to be SO_6-standard gives the result †

$$\begin{aligned}\mathrm{SO}_6\qquad &[\Delta;1]_-\times[2^21]_+\\ &=[\Delta;32]_++[\Delta;31^2]_++[\Delta;2^21]_++[\Delta;2^2]_-+2[\Delta;21]_+\\ &\quad+[\Delta;1^3]_++[\Delta;1^2]_-+[\Delta;1]_+.\end{aligned}\qquad(7.3)$$

Again $[\Delta;1]_+\times[2^21]_-$ may be found by applying (2.5) to both sides.

† The inadequacy of Fischler's Rules in such a case is made clear by the fact that in the SO_6 product $[2^21]_+\times[\Delta;1]_-$ treated by means of Rule 5c the term $[\Delta;21]_+$ arises twice: once each from the distinct intermediate representations $[\Delta;321]_+$ and $[\Delta;21^2]_+$. Rule 5d then necessitates discarding one term $[\Delta;21]_+$ in conflict with the correct result which gives a multiplicity of 2 for this term.

It is possible to rewrite some of the results (7.1) in a form more akin to (5.11) by making use of the formula

$$\sum_{\xi}\{\bar{\xi};\nu/\xi\}\cdot\{\mu\}=\sum_{\xi,\zeta}\{\overline{\xi/\zeta};(\nu/\xi)\cdot(\mu/\zeta)\}=\sum_{\eta,\zeta}\{\bar{\eta};(\nu/\zeta\eta)\cdot(\mu/\zeta)\} \qquad (7.4)$$

based on (5.2) and the link between products and quotients defined by (4.2) and (4.3). This result (7.4) then yields in conjunction with (7.1)

SO_{2k}

$$[\lambda]\times[\mu]_+=\sum_{\eta,\zeta}[\bar{\eta};(\lambda/\zeta\eta B)\cdot(\mu/\zeta)]_+ \qquad (7.5a)$$

$$[\lambda]\times[\Delta;\mu]_+=\sum_{\eta,\zeta}[\Delta;\bar{\eta};(\lambda/\zeta\eta B)\cdot(\mu/\zeta)]_+ \qquad (7.5b)$$

$$[\lambda]\times[\Box;\mu]_+=\sum_{\eta,\zeta}[\bar{\eta};(\lambda/\zeta\eta B)\cdot((\Box;\mu)/\zeta)]_+ \qquad (7.5c)$$

$$[\Delta;\lambda]_\pm\times[\Delta;\mu]_+=\sum_{\eta,\zeta}[\bar{\eta};(\lambda/\zeta\eta B)\cdot(\mu/\zeta)\cdot Q_{\pm(-)^k}]_+ \qquad (7.5d)$$

$$[\Delta;\lambda]_\pm\times[\Box;\mu]_+=\sum_{\eta,\zeta}[\Delta;\bar{\eta};(\lambda/\zeta\eta B)\cdot(\mu/\zeta)\cdot Q_{\pm(-)^k}]_+ \qquad (7.5e)$$

$$[\Box;\lambda]_\pm\times[\Box;\mu]_+=\sum_{\eta,\zeta}[\bar{\eta};(\lambda/\zeta\eta B)\cdot(\mu/\zeta)\cdot X_{\pm(-)^k}]_+. \qquad (7.5f)$$

Once again all possible cases are covered by the commutativity of Kronecker products and the use of the involutory outer automorphism, †, of SO_{2k} defined by (2.5). These results involve merely S-function products and quotients but the results must, as before, be modified in U_k and then SO_{2k}.

In a similar way the branching rules (A1) and (A2) for $SO_{2k+1}\downarrow U_k$ yield the formulae

SO_{2k+1}

$$[\lambda]\times[\mu]=\sum_{\xi}[\{\bar{\xi};\lambda/\xi F\}\cdot\{\mu\}] \qquad (7.6a)$$

$$[\lambda]\times[\Delta;\mu]=\sum_{\xi}[\varepsilon^{1/2}\{\bar{\xi};\lambda/\xi F\}\cdot\{\mu\}] \qquad (7.6b)$$

$$[\Delta;\lambda]\times[\Delta;\mu]=\sum_{\xi}[\{\bar{\xi};(\lambda/\xi F)\cdot Q\}\cdot\{\mu\}] \qquad (7.6c)$$

which may be written, thanks to (7.4), in the form

SO_{2k+1}

$$[\lambda]\times[\mu]=\sum_{\eta,\zeta}[\bar{\eta};(\lambda/\zeta\eta F)\cdot(\mu/\zeta)] \qquad (7.7a)$$

$$[\lambda]\times[\Delta;\mu]=\sum_{\eta,\zeta}[\Delta;\bar{\eta};(\lambda/\zeta\eta F)\cdot(\mu/\zeta)] \qquad (7.7b)$$

$$[\Delta;\lambda]\times[\Delta;\mu]=\sum_{\eta,\zeta}[\bar{\eta};(\lambda/\zeta\eta F)\cdot Q\cdot(\mu/\zeta)]. \qquad (7.7c)$$

Similarly using the branching rule (A3) for $Sp_{2k}\downarrow U_k$ gives

Sp_{2k}

$$\langle\lambda\rangle\times\langle\mu\rangle=\sum_{\xi}\langle\{\bar{\xi};\lambda/\xi D\}\cdot\{\mu\}\rangle \qquad (7.8)$$

and then from (7.4)

Sp_{2k}

$$\langle\lambda\rangle\times\langle\mu\rangle=\sum_{\eta,\zeta}\langle\bar{\eta};(\lambda/\zeta\eta D)\cdot(\mu/\zeta)\rangle. \qquad (7.9)$$

The results (7.9), (7.7a), (7.7b) and (7.7c) should be compared with the results (5.7), (5.11), (5.16c) and (5.16d) respectively. Of these there is no doubt that (5.7), (5.11) and (5.16c) are the simplest, whilst (7.7c) or its precursor (7.6c), has some advantage over (5.16d) because the cut-off in the series Q is reached more quickly in U_k than in SO_{2k+1}.

The final formulae of this section, (7.5), (7.7) and (7.9), have been expressed in such a way that the application of the modification rules of SO_{2k}, SO_{2k+1} and Sp_{2k}, respectively, involve in each case the conversion of either $(\bar{\eta}; \nu)$ or $(\Delta; \bar{\eta}; \nu)$ to standard form. This may be done in accordance with the rules of table 5. These are such that if k is large compared with the number of parts of ν the only non-vanishing contributions will be those for which $h = 0$. These special cases are explicitly displayed in table 5, from which it can be seen that the modification of (7.7a), (7.7b) and (7.9) is effected by dropping the summation over η, deleting $\bar{\eta}$ and replacing η by E, C and C respectively. The use of the identities (4.9) then immediately leads back to the results (5.11), (5.16c) and (5.7).

This simplification cannot be applied to (7.7c) because the infinite series Q leads to terms for which k is not large compared with the number of parts of the relevant partition ν for any finite k.

Similarly (7.5a) may be modified in the same way replacing η by A and using (4.9) to recover (5.11). However, this is only true for k large compared with the number of parts of λ and μ, so that the subscripts $\pm$ are irrelevant and may be dropped. Unfortunately none of the other formulae of (7.5) may be simplified. Progress is frustrated by either the presence of the infinite series $Q_\pm$ and $X_\pm$ or the sign changes $\pm \to \mp$ contained in the rules of table 5 appropriate to SO_{2k}, or both! Thus (7.1) or (7.5) remains the last word for Kronecker products of irreps of SO_{2k}.

A final summary of the best results for each of the classical groups is given in table 8.

It should be pointed out that the results arrived at in this section are related to but distinct from those of Girardi *et al* (1981a, b). The derivation of their remarkable results is quite different and involves yet another method of avoiding over-counting by eliminating unwanted terms by means of a modification procedure. This depends upon making a careful distinction between 'allowed' and 'non-allowed' terms. Once again the infinite S-function series (4.5) appear in the final formulae. Moreover, the special cases of modification rules given in table 5 govern the formation of these formulae.

8. Kronecker products for the exceptional groups

The final task is the application of the branching rule method to the evaluation of Kronecker products of irreps of the exceptional groups through the use of (6.10). It was precisely this application which motivated the original development of this technique (King 1981). Its worth depends upon the extent to which the required branching rules are known.

Fortunately extensive tables are available (King and Al-Qubanchi 1978, Wybourne and Bowick 1977, Wybourne 1979, McKay and Patera 1981). These should be adequate for the forseeable future, particularly as the branching rule is only required in (6.10) for one of the irreps of the exceptional group G. Of course for the simplest

Table 8. Kronecker products for the classical groups.

G	$\boldsymbol{\lambda}_G \times \boldsymbol{\mu}_G = \sum_{\nu_G} K^{\nu_G}_{\lambda_G \mu_G} \nu_G$
U_n	$\{\bar{\mu}; \lambda\} \times \{\bar{\rho}; \nu\} = \sum_{\sigma,\tau} \{\overline{(\mu/\sigma) \cdot (\rho/\tau)}; (\lambda/\tau) \cdot (\nu/\sigma)\}$
SU_n	$\{\lambda\} \times \{\mu\} = \{\lambda \cdot \mu\}$
O_{2k+1}	$[\lambda] \times [\mu] = \sum_{\zeta} [(\lambda/\zeta) \cdot (\mu/\zeta)]$
	$[\lambda] \times [\Delta; \mu] = \sum_{\zeta} [\Delta; (\lambda/\zeta Q^*) \cdot (\mu/\zeta)]$
	$[\Delta; \lambda] \times [\Delta; \mu] = \sum_{\zeta} \frac{1}{2}[(\lambda/\zeta) \cdot (\mu/\zeta) \cdot Q^*]$
SO_{2k+1}	$[\lambda] \times [\mu] = \sum_{\zeta} [(\lambda/\zeta) \cdot (\mu/\zeta)]$
	$[\lambda] \times [\Delta; \mu] = \sum_{\zeta} [\Delta; (\lambda/\zeta Q) \cdot (\mu/\zeta)]$
	$[\Delta; \lambda] \times [\Delta; \mu] = \sum_{\zeta} \frac{1}{2}[(\lambda/\zeta) \cdot (\mu/\zeta) \cdot Q] = \sum_{\eta\zeta} [\bar{\eta}; (\lambda/\zeta\eta F) \cdot (\mu/\zeta) \cdot Q]$
Sp_{2k}	$\langle\lambda\rangle \times \langle\mu\rangle = \sum_{\zeta} \langle(\lambda/\zeta) \cdot (\mu/\zeta)\rangle$
O_{2k}	$[\lambda] \times [\mu] = \sum_{\zeta} [(\lambda/\zeta) \cdot (\mu/\zeta)]$
	$[\lambda] \times [\Delta; \mu] = \sum_{\zeta} [\Delta; (\lambda/\zeta Q) \cdot (\mu/\zeta)]$
	$[\Delta; \lambda] \times [\Delta; \mu] = \sum_{\zeta} [(\lambda/\zeta) \cdot (\mu/\zeta) \cdot Q]$
SO_{2k}	$[\lambda] \times [\mu] = \sum_{\zeta} [(\lambda/\zeta) \cdot (\mu/\zeta)]$
	$[\lambda] \times [\Delta; \mu]_+ = \sum_{\eta,\zeta} [\Delta; \bar{\eta}; (\lambda/\zeta\eta B) \cdot (\mu/\zeta)]_+$
	$[\lambda] \times [\mu]_+ = \sum_{\eta,\zeta} [\bar{\eta}; (\lambda/\zeta\eta B) \cdot (\mu/\zeta)]_-$
	$[\Delta; \lambda]_\pm \times [\Delta; \mu]_+ = \sum_{\eta,\zeta} [\bar{\eta}; (\lambda/\zeta\eta B) \cdot (\mu/\zeta) \cdot Q_{\pm(-)^k}]_+$
	$[\Delta; \lambda]_\pm \times [\Box; \mu]_- = \sum_{\eta,\zeta} [\Delta; \bar{\eta}; (\lambda/\zeta\eta B) \cdot (\mu/\zeta) \cdot Q_{\pm(-)^k}]_+$
	$[\Box; \lambda]_\pm \times [\Box; \mu]_- = \sum_{\eta,\zeta} [\bar{\eta}; (\lambda/\zeta\eta B) \cdot (\mu/\zeta) \cdot X_{\pm(-)^k}]_+$

case the branching rule is completely known; that is the case of $G_2 \downarrow SU_3$ (Fronsdal 1962, King and Al-Qubanchi 1978).

The implementation of (6.10) involves a three-stage algorithm. Stage 1 provides the branching of $\boldsymbol{\lambda}_G$ to give $\boldsymbol{\sigma}_H$ along with the branching multiplicities. Stage 2 involves the addition of $\boldsymbol{\delta}$ to $\boldsymbol{\mu}_G$ and the evaluation of the product $\boldsymbol{\sigma}_H \times (\boldsymbol{\mu}_G + \boldsymbol{\delta})$ of irreps of H to give $\boldsymbol{\rho}_H$ in H-standard form. Stage 3 gives the final result through the subtraction of $\boldsymbol{\delta}$ from $\boldsymbol{\rho}_H$ and the subsequent G-standardisation of $(\boldsymbol{\rho}_H - \boldsymbol{\delta})_G$.

For example, in the case of the product $(21) \times (21)$ in G_2 the first stage gives

$$G_2 \downarrow SU_3 \qquad (21) \downarrow \{21\} + \{1\} + \{1^2\}. \tag{8.1}$$

The second stage yields, with the addition of $\boldsymbol{\delta} = (10)$,

$$\begin{aligned} \mathrm{SU}_3 \qquad & (\{21\}+\{1\}+\{1^2\})\times\{31\} \\ & =\{52\}+\{43\}+\{42\}+\{41\}+\{4\}+\{32\}+2\{31\} \\ & \quad +\{3\}+\{2^2\}+\{21\}+\{2\}+\{1\}. \end{aligned} \tag{8.2}$$

Subtraction of $\boldsymbol{\delta} = (10)$ gives

$$\begin{aligned} \mathrm{G}_2 \qquad & (42)+(3^2)+(32)+(31)+(3)+(2^2)+2(21)+(2) \\ & \quad +(12)+(1^2)+(1)+(0). \end{aligned}$$

Under the modification rules of G_2:

$$(3^2)=-(31) \qquad (32)=0 \qquad (2^2)=-(21) \qquad (12)=-(1) \qquad (1^2)=0 \tag{8.3}$$

so that the final stage gives the result

$$\mathrm{G}_2 \qquad (21)\times(21)=(42)+(3)+(21)+(2)+(0) \tag{8.4}$$

in agreement with the tabulated result due to Butler (Wybourne 1970) if due account is taken of the difference in notation: the irrep (λ_1, λ_2) is denoted by $(\lambda_1-\lambda_2, \lambda_2)$ in Butler's tabulation.

Unfortunately, in this example and in many others, the addition of $\boldsymbol{\delta}$ in stage 2 and its subtraction in stage 3 are essential steps. An incorrect result is obtained if $\boldsymbol{\delta}$ is omitted altogether. This is in contrast to the classical group case.

The group F_4 has been discussed elsewhere (King 1981) using the branching rule for $\mathrm{F}_4 \downarrow \mathrm{SO}_9$ (Wybourne and Bowick 1977, McKay and Patera 1981). There is no difficulty in combining this branching with that for $\mathrm{SO}_9 \downarrow \mathrm{U}_4$ given in the appendix: (A1) and (A2). In both cases, using SO_9 or U_4, $\boldsymbol{\delta} = (2000)$ and $\boldsymbol{\delta}$ may not be omitted.

The group E_6 is slightly easier to cope with since it is possible to take advantage of the semisimple nature of the subgroup $\mathrm{SU}_2\times\mathrm{SU}_6$ and the trivial products in SU_2 to show that in this case $\boldsymbol{\delta}$ can be reduced in evaluating the product $(s:\lambda)\times(t:\mu)$ to

$$\boldsymbol{\delta} = \begin{cases} (000000) & \text{if } s-t\leqslant 0 \\ (s-t, 00000) & \text{if } 0<s-t\leqslant 10 \\ (10,00000) & \text{if } s-t>10. \end{cases} \tag{8.5}$$

By suitably ordering any pair of irreps $(s:\lambda)$ and $(t:\mu)$ it is clearly possible to omit $\boldsymbol{\delta}$ in evaluating their product.

By way of illustration consider the product $(2:1^2)\times(2:0)$ in E_6. On restriction (Wybourne and Bowick 1977) stage 1 gives

$$\mathrm{E}_6\downarrow\mathrm{SU}_2\times\mathrm{SU}_6 \qquad (2:0)\downarrow\{2\}\{0\}+\{1\}\{1^3\}+\{0\}\{21^4\}. \tag{8.6}$$

Stage 2 yields

$$\begin{aligned} \mathrm{SU}_2\times\mathrm{SU}_6 \qquad & (\{2\}\{0\}+\{1\}\{1^3\}+\{0\}\{21^4\})(\{2\}\{1^2\}) \\ & =\{4\}\{1^2\}+\{3\}(\{2^21\}+\{21^3\}+\{1^5\}) \\ & \quad +\{2\}(2\{1^2\}+\{2^31^2\}+\{2\}+\{321^3\}) \\ & \quad +\{1\}(\{1^5\}+\{21^3\}+\{2^21\})+\{0\}\{1^2\}. \end{aligned} \tag{8.7}$$

In stage 3 we obtain first of all the terms

$$(4:1^2)+(3:2^21)+(3:21^3)+(3:1^5)+2(2:1^2)+(2:2^31^2)+(2:2)$$
$$+(2:321^3)+(1:1^5)+(1:21^3)+(1:2^21)+(0:1^2).$$

These symbols are all standard in $SU_2 \times SU_6$ but must be modified in E_6 using the appropriate rules in table 4. We find in particular that

$$(3:2^21)=(2:2^31^2)=(2:321^3)=(1:21^3)=0 \qquad \text{since } h=0$$
$$(1:2^21)=-(2:1^2) \qquad \text{since } h=1.$$

Thus we finally obtain

$$E_6 \qquad (2:1^2)\times(2:0)=(4:1^2)+(3:21^3)+(3:1^5)+(2:2)+(2:1^2)+(1:1^5) \tag{8.8}$$

in agreement with the result found earlier (Wybourne and Bowick 1977).

For E_7 and E_8 the best that can be done, in general, in evaluating $(\lambda_1, \lambda_2, \ldots)\times(\mu_1, \mu_2, \ldots)$ is to take

$$\boldsymbol{\delta}=\begin{cases}(00\ldots0) & \text{if } \lambda_1-\mu_1+\mu_2\leqslant 0\\ (\lambda_1-\mu_1+\mu_2, 0, 0\ldots0) & \text{if } 0<\lambda_1-\mu_1+\mu_2\leqslant d\\ (d, 0\ldots0) & \text{if } \lambda_1-\mu_1+\mu_2>d\end{cases} \tag{8.9}$$

with $d=10$ and 21 for E_7 and E_8 respectively. In the case of the product $(21)\times(21)$ in E_8 for example this gives $\boldsymbol{\delta}=(10000000)$. In stage 1 (Wybourne and Bowick 1977)

$$E_8\downarrow SU_9 \qquad (21)\downarrow\{21\}+\{2^71\}+\{2^21^5\}+\{2^41^4\}+\{21^4\}+\{21^7\} \tag{8.10}$$

and in stage 2

$$\begin{aligned}SU_9 \qquad &(\{21\}+\{2^71\}+\{2^21^5\}+\{2^41^4\}+\{21^4\}+\{21^7\})\times\{31\}\\ &=\{532^51\}+\{52^7\}+\{432^6\}+\{532^21^4\}+\{52^41^3\}+\{432^31^3\}\\ &\quad+\{42^51^2\}+\{531^5\}+\{52^21^4\}+2\{521^6\}+\{4321^4\}+2\{431^6\}\\ &\quad+\{42^31^3\}+3\{42^21^5\}+\{3^221^5\}+\{32^31^4\}+\{521^3\}+\{51^5\}\\ &\quad+\{431^3\}+\{42^21^2\}+2\{421^4\}+2\{41^6\}+\{3^21^4\}+\{32^21^3\}\\ &\quad+2\{321^5\}+2\{31^7\}+\{2^21^6\}+\{52\}+\{51^2\}+\{43\}+2\{421\}\\ &\quad+2\{41^3\}+\{3^21\}+\{32^2\}+2\{321^2\}+2\{31^4\}+\{2^21^3\}\\ &\quad+\{21^5\}+\{4\}+3\{31\}+\{2^2\}+2\{21^2\}+\{1\}.\end{aligned}$$

Subtracting 1 from the first component of each of these terms and applying the modification rules of table 4 for E_8 gives finally

$$\begin{aligned}E_8 \qquad (21)\times(21)&=(42^7)+(421^6)+(42)+(41^5)+(41^2)\\ &\quad+(31^6)+(3)+(21^7)+(21)+(0)\end{aligned} \tag{8.11}$$

again in agreement with the result found earlier (Wybourne and Bowick 1977).

It should be stressed that the examples given are merely intended as illustrations. Hand calculations would be hard put to extend the tabulations for products in G_2

(Wybourne 1970), in F_4 (Englefield 1981), in E_6 and E_7 (Wybourne and Bowick 1977, Englefield 1981) and in E_8 (Wybourne 1979). The merit of the technique presented here is that known branching multiplicities (Wybourne and Bowick 1977) allow for a considerable extension of Kronecker product results using a computer to implement the key algorithm we have discussed, including all the modification rules.

9. Concluding remarks

In this paper an attempt has been made to present the most efficient Schur function method of evaluating the Kronecker product of any two irreducible representations of any compact semisimple Lie group. The results, summarised in table 8, are the most efficient possible in the sense that each formula expresses the product as a sum of positive terms. The only overcounting which occurs is a consequence of the need to use modification rules, and this cannot be avoided. The succinct statement of the results, along with their derivation, owes a great deal to the symbolic manipulation of infinite series of S-functions introduced in § 4. In this way a coherent set of algorithms suitable for computer implementation has been provided.

Appendix. Branching from SO_{2k+1}, Sp_{2k} and SO_{2k} to U_k

Many branching rules for irreps of the classical groups are known (King 1975b). These results imply the validity of the following:

$$SO_{2k+1} \downarrow U_k \qquad [\lambda] \downarrow \sum_{\xi} \{\bar{\xi}; \lambda/\xi F\} \tag{A1}$$

$$[\Delta; \lambda] \downarrow \sum_{\xi} \varepsilon^{-1/2}\{\bar{\xi}; (\lambda/\xi F)\cdot Q\} = \sum_{\xi} \varepsilon^{1/2}\{\overline{\xi\cdot Q}; \lambda/\xi F\} \tag{A2}$$

$$Sp_{2k} \downarrow U_k \qquad \langle\lambda\rangle \downarrow \sum_{\xi} \{\bar{\xi}; \lambda/\xi D\} \tag{A3}$$

$$SO_{2k} \downarrow U_k \qquad [\lambda] \downarrow \sum_{\xi} \{\bar{\xi}; \lambda/\xi B\} \tag{A4}$$

$$[\Delta; \lambda] \downarrow \sum_{\xi} \varepsilon^{-1/2}\{\bar{\xi}; (\lambda/\xi B)\cdot Q\} = \sum_{\xi} \varepsilon^{1/2}\{\overline{\xi\cdot Q}; \lambda/\xi B\}. \tag{A5}$$

Although the results (A4) and (A5) apply as indicated to the restriction from SO_{2k} to U_k they do not supply the desired branching rules for the irreps $[\lambda]_\pm$ (or $[\square; \lambda]_\pm$) and $[\Delta; \lambda]_\pm$ of SO_{2k}. In order to proceed it is necessary to invoke difference character techniques (King *et al* 1981).

A crucial set of identities takes the form

$$Z\{\bar{\mu}; \lambda\} = \{\overline{\mu/Z}; \lambda Z\} \tag{A6a}$$

$$\bar{Z}\{\bar{\mu}; \lambda\} = \{\overline{\mu Z}; \lambda/Z\}. \tag{A6b}$$

These are valid for S-function series $Z = L$, M, P, Q, V and W, as may be proved readily by using (5.2) to evaluate $\{\bar{\mu}; \lambda\}\times\{\bar{\rho}; \nu\}$ in the special cases $\{\bar{\rho}; \nu\} = \{m\}$ and $\{1^m\}$, together with the definitions (4.5) and the identities (4.7) and (4.8c).

The derivation of (A5) then depends only upon the key results

$$\mathrm{SO}_{2k} \downarrow \mathrm{U}_k \qquad \Delta \downarrow \varepsilon^{-1/2} Q = \varepsilon^{1/2} \bar{Q} \tag{A7}$$

$$\mathrm{SO}_{2k} \qquad [\Delta; \lambda] = \Delta[\lambda/P] \tag{A8a}$$

$$\Delta[\lambda] = [\Delta; \lambda/Q]. \tag{A8b}$$

The analogous formulae for difference characters take the form

$$\mathrm{SO}_{2k} \downarrow \mathrm{U}_k \qquad \Delta'' \downarrow (-1)^k \varepsilon^{-1/2} L = \varepsilon^{1/2} \bar{L} \tag{A9}$$

$$\mathrm{SO}_{2k} \qquad [\Delta; \lambda]'' = \Delta''[\lambda/M] \tag{A10a}$$

$$\Delta''[\lambda] = [\Delta; \lambda/L]'' \tag{A10b}$$

and lead inexorably via (A6) with $Z = L$, to the result

$$\mathrm{SO}_{2k} \downarrow \mathrm{U}_k \qquad [\Delta; \lambda]'' \downarrow (-1)^k \sum_\xi \varepsilon^{-1/2} \{\bar{\xi}; (\lambda/\xi B) \cdot L\} = \sum_\xi \varepsilon^{1/2} \{\overline{\xi \cdot L}; \lambda/\xi B\}. \tag{A11}$$

Combining (A5) and (A11) we arrive, by virtue of (5.19*b*), at the required branching rules for spinor irreps of SO_{2k}:

$$\mathrm{SO}_{2k} \downarrow \mathrm{U}_k \qquad [\Delta; \lambda]_\pm \downarrow \sum_\xi \varepsilon^{-1/2} \{\bar{\xi}; (\lambda/\xi B) \cdot Q_{\pm(-)^k}\} = \sum_\xi \varepsilon^{1/2} \{\overline{\xi \cdot Q_\pm}; \lambda/\xi B\} \tag{A12}$$

with $Q_\pm$ defined by (4.9*a*).

For the tensor difference characters we can proceed in the same way. The fundamental branching rule is found to be

$$\mathrm{SO}_{2k} \downarrow \mathrm{U}_k \qquad \Box'' \downarrow (-1)^k \varepsilon^{-1} V = \varepsilon \bar{V} \tag{A13}$$

and the difference character identities take the form (King *et al* 1981)

$$\mathrm{SO}_{2k} \qquad [\Box; \lambda]'' = \Box''[\lambda/W] \tag{A14a}$$

$$\Box''[\lambda] = [\Box; \lambda/V]''. \tag{A14b}$$

These can be combined with (A4) and (A6) with $Z = V$ to produce

$$\mathrm{SO}_{2k} \downarrow \mathrm{U}_k \qquad [\Box; \lambda]'' \downarrow (-1)^k \sum_\xi \varepsilon^{-1} \{\bar{\xi}; (\lambda/\xi B) \cdot V\} = \sum_\xi \varepsilon \{\overline{\xi \cdot V}; \lambda/\xi B\}. \tag{A15}$$

The analogue of (A13) takes the desired form, namely

$$\mathrm{SO}_{2k} \downarrow \mathrm{U}_k \qquad \Box \downarrow \varepsilon^{-1} X = \varepsilon \bar{X}. \tag{A16}$$

However, no direct analogue of (A14) exists (King *et al* 1981) and it is not possible to use (A4) as it stands to derive an analogue of (A15). Nonetheless extensive manipulations on (A4) do indeed yield the anticipated result:

$$\mathrm{SO}_{2k} \downarrow \mathrm{U}_k \qquad [\Box; \lambda] \downarrow \sum_\xi \varepsilon^{-1} \{\bar{\xi}; (\lambda/\xi B) X\} = \sum_\xi \varepsilon \{\overline{\xi \cdot X}; \lambda/\xi B\}. \tag{A17}$$

Combining this with (A15) completes the set of branching rules for irreps of SO_{2k} with the result

$$\mathrm{SO}_{2k} \downarrow \mathrm{U}_k \qquad [\Box; \lambda]_\pm \downarrow \sum_\xi \varepsilon^{-1} \{\bar{\xi}; (\lambda/\xi B) X_{\pm(-)^k}\} = \sum_\xi \varepsilon \{\overline{\xi \cdot X_\pm}; \lambda/\xi B\} \tag{A18}$$

with $X_\pm$ defined by (4.9*b*).

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